Reduction of hypercubic lattice artifacts

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GDR “Physique subatomique et calculs sur réseau”
CPT Marseille, June 25-27, 2008

An example of hypercubic lattice artifacts

Free scalar field

$$\Delta_L(p) = \frac{1}{\widehat{p}^2 + m^2}$$

$$\widehat{p}^2 = \sum_\mu \widehat{p}_\mu^2$$

$$\widehat{p}_\mu = \frac{2}{a} \sin\left(\frac{ap_\mu}{2}\right)$$

$$p_\mu = \frac{2\pi n_\mu}{aL}$$

$$\Delta_0(p) = \frac{1}{p^2 + m^2}$$

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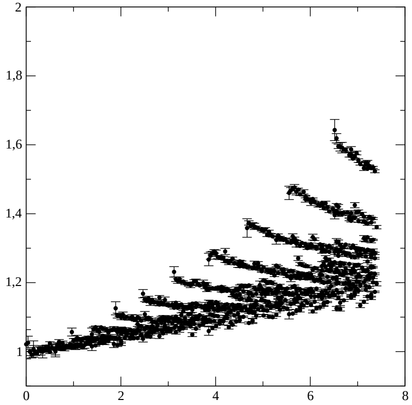

Figure: *Raw dressing function $\Delta_L(p)/\Delta_0(p)$ as a function of p^2/m^2 for a 32^4 lattice and $am = 1$ (1000 configurations).*

H4 extrapolation method

- Orsay lattice group collaboration.
- Lattice study of the Green functions of QCD:
  - Gluon, quark and ghost propagators $\longrightarrow \Lambda_{QCD}$.
  - Three-point Green functions $\longrightarrow \alpha_S$.
- Comparison with perturbation theory in ultraviolet regime $\Longrightarrow$ elimination of hypercubic artifacts from raw data.

Range of applicability of H4 extrapolation method:

F. de Soto, C.R., JHEP 0709:007,2007; arXiv:0705.3523.

Plan of the talk

The democatic method

- Apply a cylindrical cut to the raw data around the diagonal $(1,1,1,1)$ of momentum space:

$$|p_\mu - p_\nu| \leq 2\,, \qquad \forall \mu, \nu$$

- Pretend that the remaining “democratic” points fall onto a smooth “continuum” curve. fig

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Have a critical look at figures of lattice papers!

The free scalar field on the lattice

Lattice action:

$$S_L = \frac{a^4}{2}\sum_x \left\{ m^2\phi_x^2 + \sum_{\mu=1}^{4}(\nabla_\mu\phi_x)^2 \right\}, \qquad \nabla_\mu\Phi_x = \frac{\Phi_{x+a\widehat{\mu}} - \Phi_x}{a}$$

Momentum space:

$$\widetilde{S}_L = \frac{a^4}{2}\sum_p \left(m^2 + \widehat{p}^2\right)|\widetilde{\phi}_p|^2, \qquad \widehat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$$

Lattice spacing expansion: $\overline{p}^n = \sum_\mu p_\mu^n$

$$\widehat{p}^2 = \sum_\mu \widehat{p}_\mu^2 \approx p^2 - \frac{a^2}{12}\overline{p}^4 + \frac{a^4}{360}\overline{p}^6 - \frac{a^6}{20160}\overline{p}^8 + \cdots$$

Lattice scalar propagator

- Lattice spacing expansion:

$$\begin{aligned}\Delta_L(p) &= \frac{1}{\widehat{p}^2 + m^2} \\ &\approx \frac{1}{p^2+m^2} + a^2 \left\{ \frac{1}{12} \frac{\overline{p}^4}{(p^2+m^2)^2} \right\} \\ &+ a^4 \left\{ \frac{1}{72} \frac{\overline{p}^{4^2}}{(p^2+m^2)^3} - \frac{2}{8!} \frac{\overline{p}^6}{(p^2+m^2)^2} \right\} + \cdots \\ &= \overline{\Delta}_L(p^2, \overline{p}^4, \overline{p}^6, \overline{p}^8)\end{aligned}$$

- Smooth continuum extrapolation $a \to 0$ and fixed p.

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- Smooth continuum extrapolation $a \to 0$ and fixed p.
- Smooth rotational extrapolation $\overline{p}^n \to 0$, $n = 4, 6, 8$, $a \neq 0$.

Hypercubic symmetry

- Work on hypercubic lattices with discrete symmetry group $H(4)$:

$$H(4) \simeq S(4) \ltimes Z(2)^4$$

- Lattice form factor: $F_L(p) \equiv F_L(p_1, p_2, p_3, p_4)$
- $F_L(p)$ is invariant along each orbit $O(p)$ in $H(4)$:

$$O(p) \equiv \{p' = h(p), \quad \forall h \in H(4)\}$$

- All orbits of a discrete group are uniquely characterized by a finite set of *polynomial* invariants.
- For $H(4)$, the algebraically independent polynomial invariants are $p^2, \overline{p}^4, \overline{p}^6, \overline{p}^8$.
- Each orbit $O(p)$ is also characterized by a subgroup of $H(4)$, the isotropy group of p which leaves p fixed.

fig

Rotational limit .vs. continuum limit

- Want to eliminate hypercubic artifacts from F_L:
  $\longrightarrow$ Rotationally invariant observable depend only on p^2.
- Extrapolate at finite lattice spacing.
- Hence the rotational limit is not the continuum limit.
- There are still scaling violations in extrapolated observables which are $O(4)$ invariant.

Renormalizable Quantum Field Theory (I)

- Functional identity:

$$F_L(p_1, p_2, p_3, p_4) = \overline{F}_L(p^2, \overline{p}^4, \overline{p}^6, \overline{p}^8)$$

- $L \longrightarrow \infty \quad \Longrightarrow \quad \overline{F}_L \longrightarrow$ smooth function.
- Rotational limit is a smooth function of p^2:

$$\overline{F}_L(p^2, 0, 0, 0) = F_O(p^2)$$

- Smooth interpolation at finite L and fixed p^2 with $a^2 p^2 \ll 1$:

$$\overline{F}_L(p^2, \overline{p}^4, \overline{p}^6, \overline{p}^8) = \overline{F}_L(p^2, 0, 0, 0) + \overline{p}^4 \frac{\partial \overline{F}_L}{\partial \overline{p}^4}(p^2, 0, 0, 0)$$
$$+ \overline{p}^6 \frac{\partial \overline{F}_L}{\partial \overline{p}^6}(p^2, 0, 0, 0) + (\overline{p}^4)^2 \frac{\partial^2 \overline{F}_L}{\partial (\overline{p}^4)^2}(p^2, 0, 0, 0) + \cdots$$

Renormalizable Quantum Field Theory (II)

Lattice spacing expansion:

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Why not $F_L(p_1,p_2,p_3,p_4) = \widehat{F}_L(\widehat{p}_1,\widehat{p}_2,\widehat{p}_3,\widehat{p}_4)$?

- $\widehat{p}_\mu$ are the natural variables of PT in a box with PBC.
- $\widehat{F}_L(\widehat{p}^2,0,0,0)$ is a smooth function of $\widehat{p}^2$ and not of p^2.
- But no model-independent way to extract $\widehat{F}_L(\widehat{p}^2,0,0,0)$ from raw data.
- $\widehat{p}^2$ is not an $O(4)$ invariant.

A controllable model

- Use the free scalar field with mass $ma = 1$ as a controllable model to generate the raw lattice data.
- Do not use any analytical or physical information in the analysis (except the smoothness assumption and naive dimensional arguments). In particular do not use the explicit knowledge of the mass.
- The case of QCD is simpler as long as Λ_{QCD} and the quark masses are negligible in comparison to the momentum scale.

Semi-local fit

Independent extrapolation
at each p^2:

$$\Delta_L(p) = \Delta_E(p^2) + c(p^2)\overline{p}^4$$
$$\Delta_E(p^2) = \Delta_L(p^2, 0, 0, 0)$$
$$c(p^2) = \frac{c_{-1}}{p^2} + c_0 + c_1 p^2$$

(for points with one orbit)

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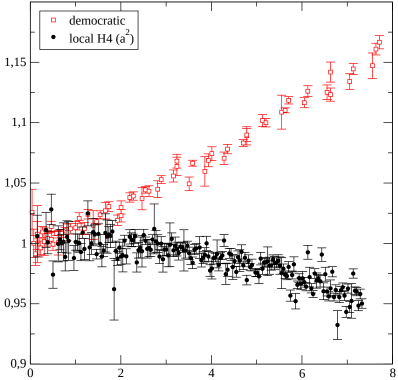

Comparison of the extrapolated dressing function $\Delta_E(p^2)/\Delta_0(p^2)$ as a function of p^2 on a 32^4 lattice ($a = m = 1$), between the democratic method (open squares) and the local H4 method (black circles) - **1000 configurations**.

Global fit

Global extrapolation up to order a^4 over some momentum window:

$$\Delta_L(p) = \Delta_E(p^2) + f_1(p^2)\overline{p}^4 + f_2(p^2)\overline{p}^6 + f_3(p^2)(\overline{p}^4)^2$$

$$f_n(p^2) = \sum_{i=-1}^{1} c_{i,n}(p^2)^{-i}$$

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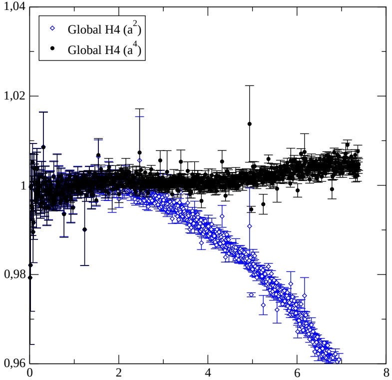

Comparison of the extrapolated dressing function $\Delta_E(p^2)/\Delta_0(p^2)$ as a function of p^2 on a 64^4 lattice ($a = m = 1$), between the global methods with $\mathcal{O}(a^2)$ artifacts (open losanges) and $\mathcal{O}(a^4)$ (black circles) - ***1000 configurations****.*

A χ^2-criterion for estimating the systematic errors

Quantitative estimation of the systematic errors as a function of lattice and sample sizes:

$$\chi^2 = \sum_{p^2=1}^{p^2_{max}} \left(\frac{\Delta_E(p^2) - \Delta_0(p^2)}{\delta\Delta_E(p^2)} \right)^2$$

For global fits, one must include the covariance matrix.

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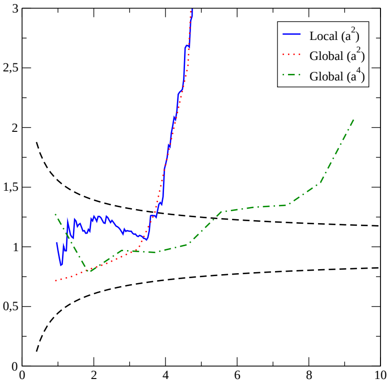
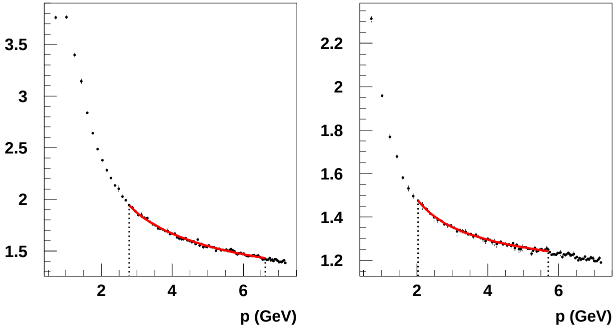


$\chi^2/d.o.f$ as a function of p^2_{max} on a 32^4 lattice ($a = m = 1$), for the local a^2 method (blue solid line), the global a^2 method (red dotted line) and the global a^4 method (green dash-dotted line). The dashed curves are the 95% CL lines.

A QCD application in the guise of conclusion



Extrapolated 32^4 *lattice data at* $\beta = 6.4$ *for* Z_3 *(left) and* $\widetilde{Z}_3$ *(right). The solid line is the fit at four-loop order in the* $\overline{MS}$ *scheme. The vertical dotted lines delimit the window of each fit.*

Anistropic lattices $L^3 \times T$

- Cubic symmetry O_h within each timeslice.
- Extrapolation towards a 3d-rotationally invariant limit for each t (or $E \equiv p_0$).
- $F_{L,T}(p) \longrightarrow \overline{F}_{L,T}(\vec{p}^2, E^2) \underset{L,T\to\infty}{\longrightarrow} F_O(p^2)$

A few facts about (hyper)cubic groups

Cubic group

- O_h has 10 conjugacy classes.
- $O_+ = O_h \cap SO(3)$ has 5 conjugacy classes.

Hypercubic group

- H_4 has 20 conjugacy classes.
- $H_4^+ = H_4 \cap SO(4)$ has 13 conjugacy classes.