Applications of Chiral Perturbation theory to lattice QCD (II)

Adapted and extended from [hep-lat/0607016]

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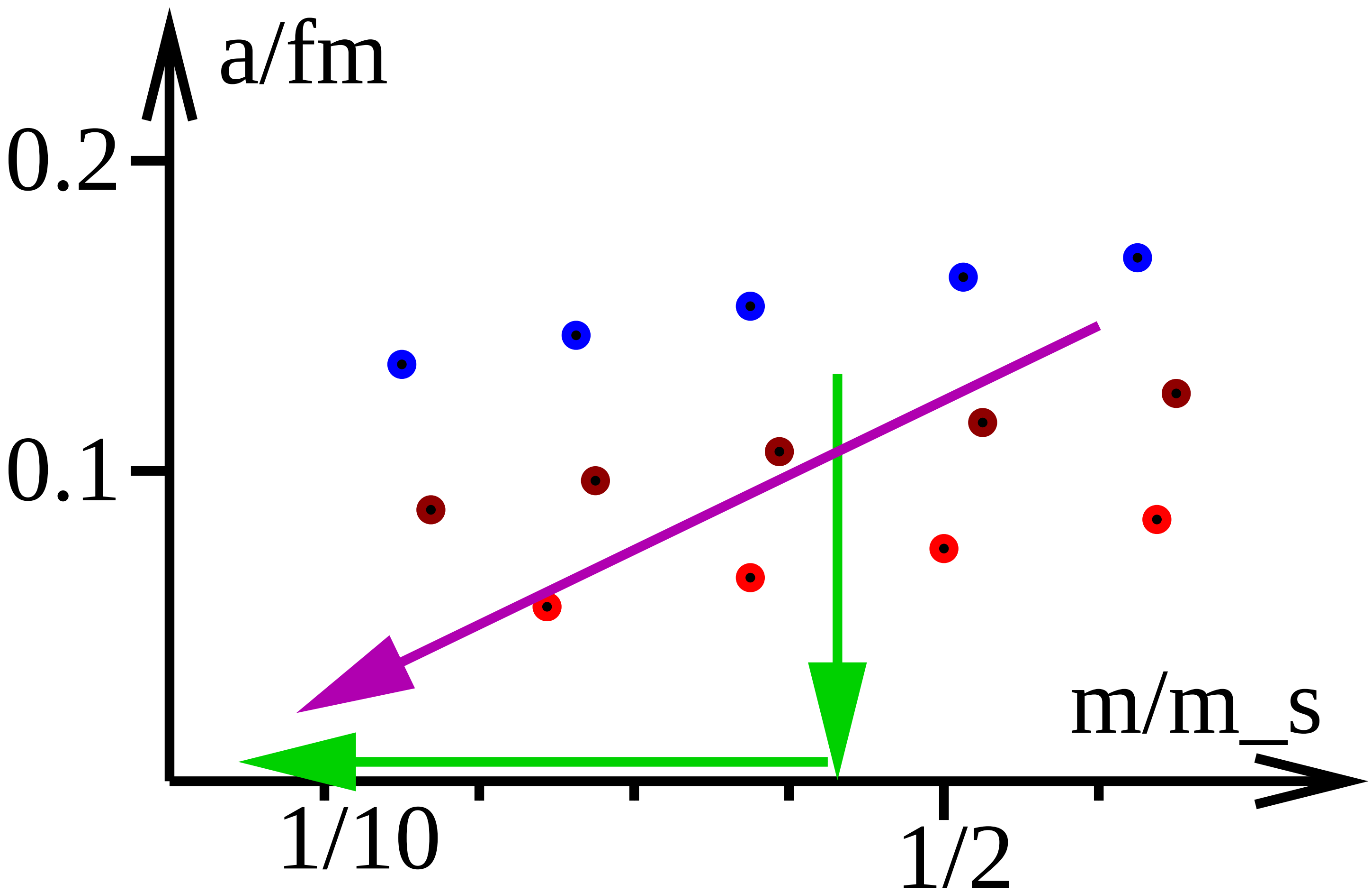
Outline of Lecture 2

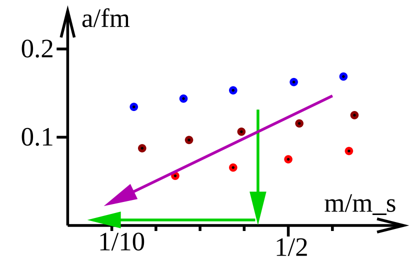
- Incorporating discretization errors into χPT
  - Why is this useful?
  - General two-step strategy and power counting
- Application to Wilson & twisted mass fermions
  - Wilson and twisted-mass lattice fermions
  - Symanzik effective action
  - Mapping Symanzik action into χPT
  - Results for $m_q \sim a\Lambda^2_{\rm QCD}$
  - Defining the twist angle
  - Results for $m_q \sim a^2\Lambda^3_{\rm QCD}$

Why incorporate discretization errors?

Why do a combined $a \to 0$ and $m \to m_{\rm phys}$ extrapolation?
Why not extrapolate $a \to 0$ and then use continuum χPT?



Why incorporate discretization errors?

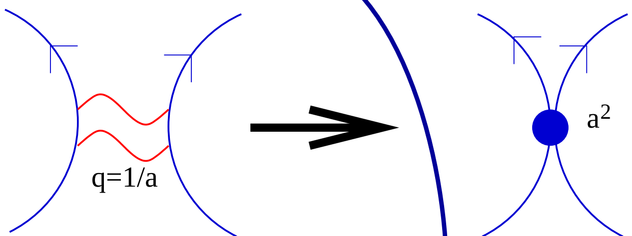


- **Incorporates symmetry relations between discretization errors**
  - **Limited number of new LECs**
- **Incorporates non-analyticities due to PGB loops**
  - **tmχPT, SχPT:** $\mathbf{m_\pi^2 \sim m_q[1 + (m_q + a^2)\ln(m_q + a^2) + \ldots]}$
  - **"a^2" means "up to logs", so not all non-analyticities are included**
- **Avoids need to work at constant physical parameters—can just fit**
- **Gives framework for understanding symmetry breaking due to discretization**
  - **Chiral symmetry broken with Wilson fermions**
  - **Chiral and flavor symmetry broken with tm fermions**
  - **Taste symmetry broken with staggered fermions**
- **Predicts phase structure when** $\mathbf{m_q \sim a^2 \Lambda_{QCD}^3}$

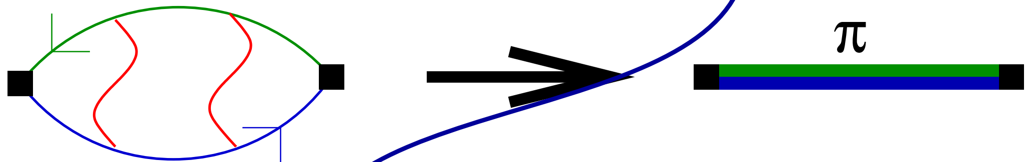
General strategy

Proceed in two steps: [Sharpe & Singleton]

Lattice Lagrangian:
Wilson, tm, staggered



Continuum effective Lagrangian:
continuum quark-level theory including explicit nonzero a effects [Symanzik]



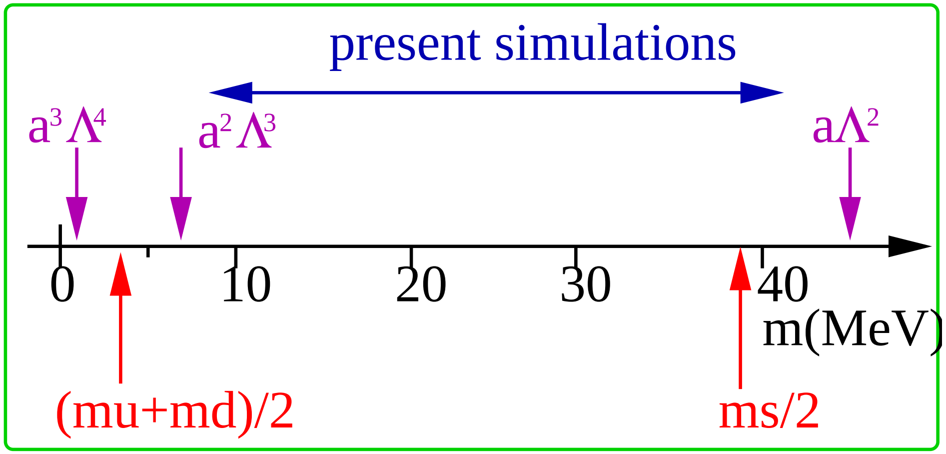
Chiral Lagrangian:
continuum χPT plus effects of additional operators induced by discretization

General comment

- Strange that UV effects impact IR effective FT?
- **NO!**
  - Conceptually same as weak-interaction effects on strongly-interacting particles:
    - Effects are small ($\propto \Lambda_{\rm QCD}^2/M_W^2$), but dominant for some processes (weak decays)
  - Here, discretization errors important when they break a symmetry that is important for IR physics—chiral symmetry.

Power counting

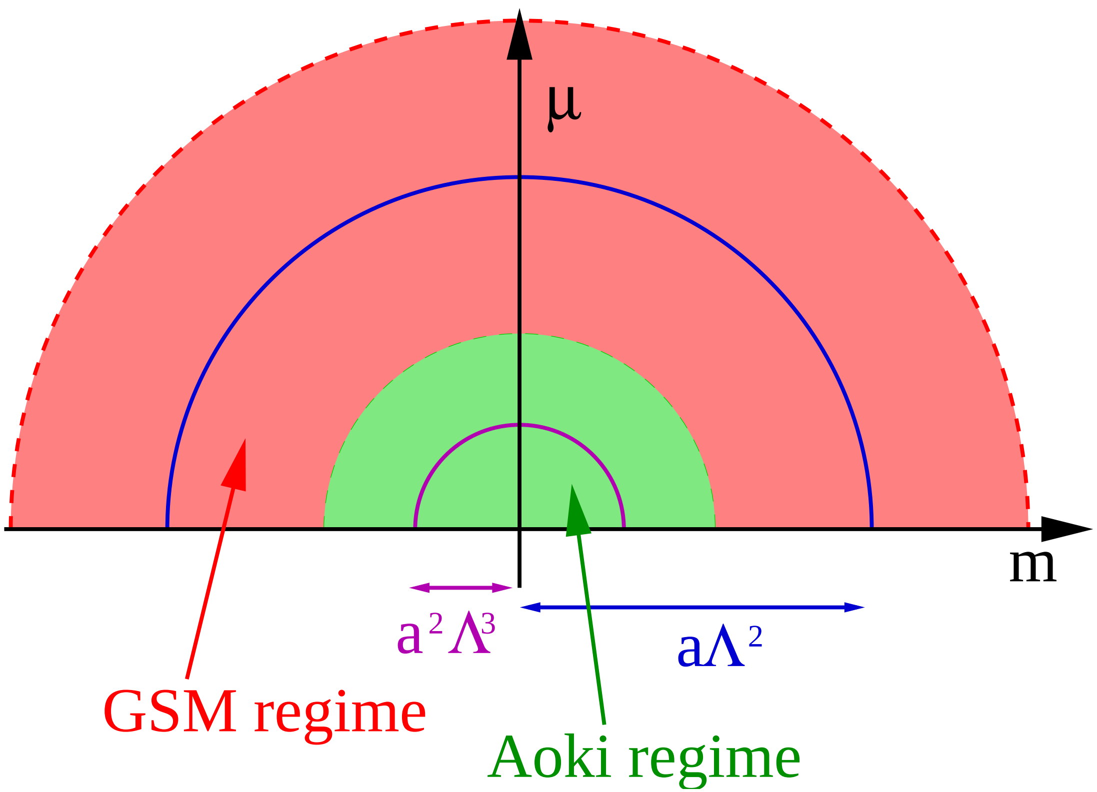
- In χPT expand in $p^2/\Lambda_\chi^2 \sim m_{\mathbf{PGB}}^2/\Lambda_\chi^2 \sim m_q/\Lambda_{\rm QCD}$
  - How does $(a\Lambda_{\rm QCD})^n$ compare?
- Compare m_q to $a\Lambda_{\rm QCD}^2$, $a^2\Lambda_{\rm QCD}^3, \ldots$
  - If $a^{-1} = 2$ GeV and $\Lambda_{\rm QCD} = 300$ MeV, then



- Appropriate power counting is $a^2\Lambda_{\rm QCD}^3 \lesssim m_q \lesssim a\Lambda_{\rm QCD}^2$
- **LESSON: $O(a)$ effects MUST BE REMOVED, and $O(a^2)$ understood**

Power counting terminology

- **Generic Small Mass (GSM) regime**: $a\Lambda_{\rm QCD}^2 \lesssim m_q \ll \Lambda_{\rm QCD}$
  - Includes $a\Lambda_{\rm QCD}^2 \ll m_q$ but *not* $m_q \ll a\Lambda_{\rm QCD}^2$
- **Aoki regime**: $m_q \lesssim a^2\Lambda_{\rm QCD}^3$
  - Includes $m_q \ll a^2\Lambda_{\rm QCD}^3$



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Selected references for twisted mass LQCD

- R. Frezzotti, P. A. Grassi, S. Sint and P. Weisz, "A local formulation of lattice QCD without unphysical fermion zero modes," Nucl. Phys. Proc. Suppl. **83**, 941 (2000) [arXiv:hep-lat/9909003].
- R. Frezzotti, P. A. Grassi, S. Sint and P. Weisz [Alpha collaboration], "Lattice QCD with a chirally twisted mass term," JHEP **0108**, 058 (2001) [arXiv:hep-lat/0101001].
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- S. Sint, "Lattice QCD with a chiral twist," arXiv:hep-lat/0702008.
- A. Shindler, "Twisted mass lattice QCD," Phys. Rept. **461**, 37 (2008) [arXiv:0707.4093 [hep-lat]].

What are twisted mass fermions?

- In continuum simply QCD with $\mathbf{M} \neq \mathbf{M}^\dagger$

$$\mathcal{L}^{q}_{\mathbf{QCD}} = \overline{\mathbf{Q}}_{\mathbf{L}} \not{D} \mathbf{Q}_{\mathbf{L}} + \overline{\mathbf{Q}}_{\mathbf{R}} \not{D} \mathbf{Q}_{\mathbf{R}} + \overline{\mathbf{Q}}_{\mathbf{L}} \mathbf{M} \mathbf{Q}_{\mathbf{R}} + \overline{\mathbf{Q}}_{\mathbf{R}} \mathbf{M}^\dagger \mathbf{Q}_{\mathbf{L}}$$

  - $\overline{\mathbf{Q}}_{\mathbf{L,R}} = \overline{\mathbf{Q}}_{\mathbf{L,R}}(\mathbf{1} \mp \gamma_5)/\mathbf{2}$, $\mathbf{Q}_{\mathbf{L,R}} = [(\mathbf{1} \pm \gamma_5)/\mathbf{2}]\mathbf{Q}_{\mathbf{L,R}}$

- Can diagonalize M with an $SU(3)_L \times SU(3)_R$ rotation: $\mathbf{M} \longrightarrow \mathbf{U}_{\mathbf{L}} \mathbf{M} \mathbf{U}^\dagger_{\mathbf{R}}$, so twisting the mass is a change of basis
  - Condensate is axially rotated, but physics is unchanged
  - Apparent breaking of parity and flavor is illusory

- Example of most interest has two degenerate flavors of mass $\mathbf{m_q}$

$$\mathbf{M} = \mathbf{m_q} \mathbf{e}^{\mathbf{i}\tau_3 \omega} = \mathbf{m_q}(\cos\omega + \mathbf{i}\sin\omega\tau_3) \equiv \mathbf{m} + \mathbf{i}\mu\tau_3$$

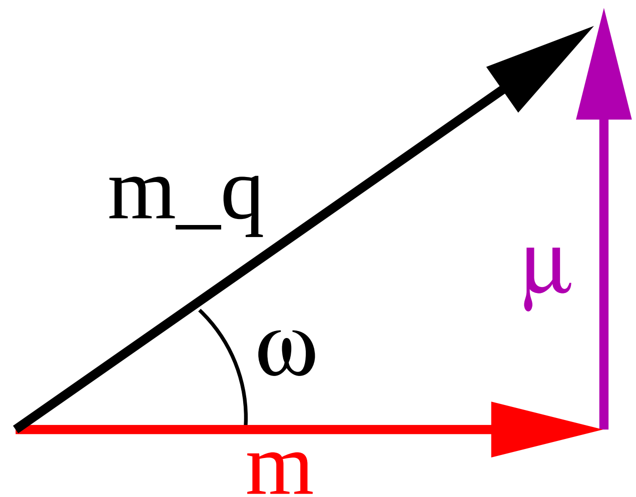
  - More familiar as

$$\overline{\mathbf{Q}}_{\mathbf{L}} \mathbf{M} \mathbf{Q}_{\mathbf{R}} + \overline{\mathbf{Q}}_{\mathbf{R}} \mathbf{M}^\dagger \mathbf{Q}_{\mathbf{L}} = \overline{\mathbf{Q}}(\mathbf{m} + \mathbf{i}\mu\gamma_5\tau_3)\mathbf{Q}$$

Twisted mass QCD

- "Geometry" of tmQCD:

$$\mathcal{L}^{\mathbf{q}}_{\mathrm{tmQCD}} = \overline{\mathbf{Q}}\not{D}\mathbf{Q} + \overline{\mathbf{Q}}(\mathbf{m} + \mathbf{i}\mu\gamma_{\mathbf{5}}\tau_{\mathbf{3}})\mathbf{Q}$$



- ω is redundant, and can use this freedom to pick a better lattice action
  - "Maximal twist", i.e. $\omega = \pm\pi/2$ ($\Rightarrow m = 0$) leads to automatic absence of $O(a)$ terms [Frezzotti & Rossi]

Discretizing (twisted mass) QCD

$$S_{\rm tmQCD} = S_{\rm glue} + \int_x \overline{Q} \not{D} Q + \overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L$$

$$\downarrow$$

$$S^{\rm lat}_{\rm tmQCD} = S^{\rm lat}_{\rm glue} + a^4 \sum_x \overline{\psi}_l D_W \psi_l + \overline{\psi}_{l,L} M \psi_{l,R} + \overline{\psi}_{l,R} M^\dagger \psi_{l,L}$$

- Uses Wilson's doubler-free derivative:

$$\not{D} \longrightarrow D_W = \frac{1}{2} \sum_\mu \gamma_\mu (\nabla^*_\mu + \nabla_\mu) - \frac{r}{2} \sum_\mu (\nabla^*_\mu \nabla_\mu)$$

- D_W breaks chiral symmetry
  - $\Rightarrow$ M and $U_L M U_R^\dagger$ describe different theories on the lattice
- Full fermion matrix $D_W + M P_R + M^\dagger P_L$ has real positive determinant (and is thus useful in practice) only for special M
  - e.g. standard twisted mass $M = m + i\mu\tau_3$ for any m, μ

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References for effective action

- K. Symanzik, “Continuum Limit And Improved Action In Lattice Theories. 1. Principles And Phi**4 Theory,” Nucl. Phys. B **226**, 187 (1983); “Continuum Limit And Improved Action In Lattice Theories. 2. O(N) Nonlinear Sigma Model In Perturbation Theory,” Nucl. Phys. B **226**, 205 (1983).
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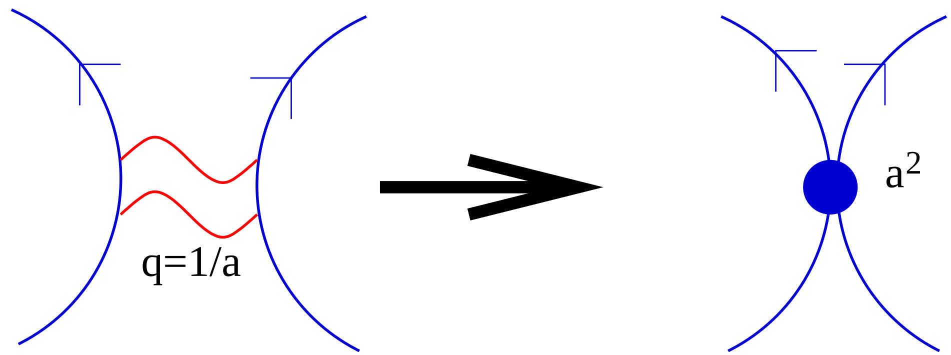
Symanzik EFT



- Integrate out high-momentum quarks and gluons ($p \sim 1/a$), obtain a local EFT describing low-momentum modes ($p \ll 1/a$)

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm tmQCD} + a\mathcal{L}^{(5)} + a^2\mathcal{L}^{(6)} + \dots$$

  - Regularize with continuum regulator or finer lattice
  - Factors of a explicit
  - "a" means $\sim a(1 + g[a]^2 \ln a + \dots)$
  - $\mathcal{L}^{(5,6,\dots)}$ contain all operators allowed by *lattice symmetries*
- $\mathcal{L}_{\rm eff}$ gives discretization errors to **all** correlation functions
  - Holds to all orders in PT (where can calculate $\mathcal{L}^{(5,6,\dots)}$) [Symanzik]
  - Demonstrates validity of EFT directly in Euclidean space

Symanzik EFT and Improvement

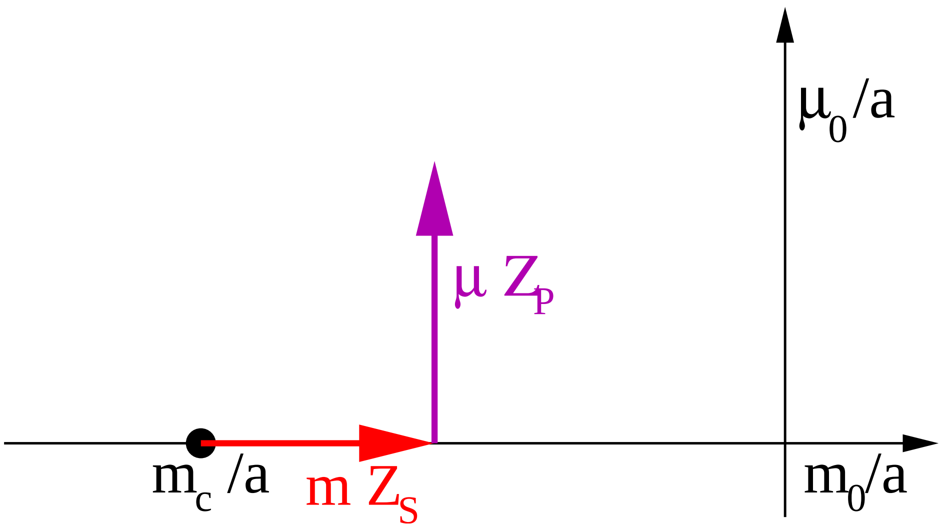
$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{tmQCD}} + a\mathcal{L}^{(5)} + a^2\mathcal{L}^{(6)} + \ldots$$

- [Symanzik] also showed that can systematically remove $\mathcal{L}^{(5,6,\ldots)}$ by adding corresponding terms to $\mathcal{L}^{\text{lat}}$: **IMPROVEMENT**
  - In practice, only $\mathcal{L}^{(5)}$ has been removed
    - **e.g. NP $O(a)$ improved Wilson fermions**
  - Attractive approach—disadvantage for matrix elements is that each operator needs separate $O(a)$ improvement
- We keep both $\mathcal{L}^{(5)}$ and $\mathcal{L}^{(6)}$
  - This is what is done in tmLQCD
    - **Why? Will see that $O(a)$ improvement automatic for $m \approx 0$**
  - Can remove $\mathcal{L}^{(5)}$ by hand to encompass improved Wilson fermions

Leading term in continuum limit for tmLQCD

$$\mathcal{L}^{\rm lat}_{\rm tmQCD} = \mathcal{L}^{\rm lat}_{\rm glue} + \overline{\psi}_l \left(D_W + m_0 + i\gamma_5\tau_3\mu_0\right)\psi_l$$

- Wilson term $\nabla^\star_\mu \nabla_\mu$ mixes with the identity
  - $\Rightarrow$ usual additive renormalization of m_0: $m = Z_S^{-1}(m_0 - m_c)/a$
- μ_0 is multiplicatively renormalized, like m_q in continuum:
  $$\mu = Z_P^{-1}\mu_0/a$$
- Thus leading term in Symanzik expansion is (by construction)
  $$\mathcal{L}^q_{\rm tmQCD} = \overline{Q}\not{D}Q + \overline{Q}(m + i\mu\gamma_5\tau_3)Q$$



Symmetries of tmLQCD

$$S^{\rm lat}_{\rm tmQCD} = S^{\rm lat}_{\rm glue} + a^4 \sum_x \overline{\psi}_l \left(D_W + m_0 + i\gamma_5\tau_3\mu_0\right)\psi_l$$

- $\mathcal{L}^{(5)}, \ldots$ are constrained by the symmetries of tm**L**QCD
- These are the standard symmetries: gauge invariance, lattice rotations and translations, C, fermion number, reflection positivity
- But only a subgroup of flavor $SU(2)$ and parity survive if $\mu_0 \neq 0$:
  - $U(1) \in SU(2)$ with generator τ_3
    - forbids $\bar\psi\tau_{1,2}\psi$ terms in $\mathcal{L}_{\rm tmQCD}$
  - $\mathcal{P}_F^{1,2}$: parity plus discrete flavor rotation
    - $\psi_l(x) \to \gamma_0(i\tau_{1,2})\psi_l(x_P)$, $\bar\psi_l(x) \to \bar\psi_l(x_P)(-i\tau_{1,2})\gamma_0$
    - forbid $\bar\psi\gamma_5\psi$, $\widetilde{F}_{\mu\nu}F_{\mu\nu}$, $\bar\psi\tau_3\psi$
  - $\widetilde{\mathcal{P}}$: parity combined with $[\mu_0 \to -\mu_0]$
    - requires $\bar\psi\tau_3\gamma_5\psi$ to come with factor $\mu_0 \propto \mu$
- Flavor-parity breaking for $a \neq 0$ are price for automatic $O(a)$ improvement

Symanzik action for tmLQCD [Sharpe & Wu]

- Straightforward extension of analysis for Wilson fermions [Luscher et al]

$$\begin{aligned}\mathcal{L}^{(5)} &= b_1 \bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi + b_2\bar\psi(\not{D} + m + i\gamma_5\tau_3\mu)^2\psi \\ &+ b_3 m\bar\psi(\not{D} + m + i\gamma_5\tau_3\mu)\psi + b_4 m\mathcal{L}_{\rm glue} + b_5 m^2\bar\psi\psi \\ &+ b_6\mu\bar\psi\{(\not{D} + m + i\gamma_5\tau_3\mu), i\gamma_5\tau_3\}\psi + b_7\mu^2\bar\psi\psi\end{aligned}$$

  - Write in terms of continuum masses m, μ rather than bare masses
  - b_i are real (refl. pos.) and depend on $g^2[a]$ and $\ln a$
  - $b_{6,7}$ are "new" compared to Wilson case (vanish when $\mu \to 0$)
  - Many terms forbidden by lattice symmetries, e.g.
    - $\widetilde{\mathcal{P}}$ forbids: $m\mu\bar\psi\psi$, $m^2\bar\psi i\gamma_5\tau_3\psi$
    - $\widetilde{\mathcal{P}}$ requires twisted Pauli term $\bar\psi\sigma_{\mu\nu}F_{\mu\nu}\tau_3\psi$ to have factor of μ and thus appear in $\mathcal{L}^{(6)}$

Simplifying $\mathcal{L}^{(5)}$

- Simplify using change of variables (equivalent to using LO eqns. of mtn.)
  - e.g. $\psi \to [1 + O(a)\not{D} + O(a)m + O(a)i\gamma_5\tau_3\mu]\psi$
  - Convenient but not essential (so don't have to worry about what happens to sources)

$$\begin{aligned}\mathcal{L}^{(5)} \quad = \quad & b_1\bar{\psi}i\sigma_{\mu\nu}F_{\mu\nu}\psi + b_2\bar{\psi}(\not{D} + m + i\gamma_5\tau_3\mu)^2\psi \\ & + b_3 m\bar{\psi}(\not{D} + m + i\gamma_5\tau_3\mu)\psi + b_4 m\mathcal{L}_{\rm glue} + b_5 m^2\bar{\psi}\psi \\ & + b_6\mu\bar{\psi}\{(\not{D} + m + i\gamma_5\tau_3\mu), i\gamma_5\tau_3\}\psi + b_7\mu^2\bar{\psi}\psi\end{aligned}$$

- b_4 leads to am dependence of $g^2_{\rm eff}$ and thus of a
- $b_{5,7}$ imply $m_{\rm phys} = m[1 + O(am)] + O(a\mu^2)$
- These effects are present, but are NNLO if use GSM power counting:

$$m/\Lambda_{\rm QCD} \sim \mu/\Lambda_{\rm QCD} \sim a\Lambda_{\rm QCD}$$

- We will keep up to quadratic order in these small parameters
  - Other power counting choices possible, e.g. [Aoki, Aoki et al]

Conclusion for $\mathcal{L}^{(5)}$

- At NLO in our power counting, only need Pauli term

$$\begin{aligned}\mathcal{L}^{(5)}_{\rm NLO} \quad = \quad & b_1 \bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi + b_2\bar\psi(\not{D} + m + i\gamma_5\tau_3\mu)^2\psi \\ & + b_3 m\bar\psi(\not{D} + m + i\gamma_5\tau_3\mu)\psi + b_4 m\mathcal{L}_{\rm glue} + b_5 m^2\bar\psi\psi \\ & + b_6\mu\bar\psi\{(\not{D} + m + i\gamma_5\tau_3\mu), i\gamma_5\tau_3\}\psi + b_7\mu^2\bar\psi\psi\end{aligned}$$

  - Same $\mathcal{L}^{(5)}$ as for Wilson fermions
  - Breaks chiral symmetry even when $m, \mu \to 0$

Results for $\mathcal{L}^{(6)}$

- Gluonic terms [Lüscher & Wiesz]

$$\mathcal{L}^{(6)}_{\rm glue} \sim \mathrm{Tr}(D_\mu F_{\rho\sigma} D_\mu F_{\rho\sigma}) + \mathrm{Tr}(D_\mu F_{\mu\sigma} D_\rho F_{\rho\sigma}) + \underbrace{\mathrm{Tr}(D_\mu F_{\mu\sigma} D_\mu F_{\mu\sigma})}_{\textbf{Lorentz violating}} + \underbrace{(m^2, \mu^2)\mathrm{Tr}(F_{\mu\nu} F_{\mu\nu})}_{O(a^2 m^2, a^2\mu^2)\ \textbf{so NNNLO}}$$

- Fermionic terms (generalizing Wilson result [Sheikholeslami & Wohlert])

$$\mathcal{L}^{(6)}_q \sim \underbrace{\bar\psi D^3_\mu \gamma_\mu \psi}_{\textbf{Lorentz violating}} + \underbrace{\bar\psi D_\mu \not{D} D_\mu \gamma_\mu \psi}_{O(a^2)\ \textbf{so NLO}} + \underbrace{(\bar\psi\psi)^2 + (\bar\psi\gamma_\mu\psi)^2 + \dots}_{O(a^2)\ \textbf{so NLO}}$$

$$+ \underbrace{m\bar\psi \not{D}^2 \psi + \mu\bar\psi\not{D}^2 i\gamma_5\tau_3\psi}_{O(a^2 m, a^2\mu)\ \textbf{so NNLO}} + \underbrace{m\bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi + \mu\bar\psi\sigma_{\mu\nu}F_{\mu\nu}\gamma_5\tau_3\psi}_{O(a^2 m, a^2\mu)\ \textbf{so NNLO}}$$

$$+ \underbrace{(m^2, \mu^2)\bar\psi\not{D}\psi + m\mu\bar\psi\not{D} i\gamma_5\tau_3\psi}_{O(a^2 m^2),\ \textbf{etc. so NNNLO}}$$

$$+ \underbrace{(m^3, m\mu^2)\bar\psi\psi + (\mu^3, \mu m^2) i\gamma_5\tau_3\psi}_{O(a^2 m^3),\ \textbf{etc. so NNNNLO}}$$

NLO part of $\mathcal{L}^{(6)}$

- Final NLO result is the same as for Wilson fermions:

$$\begin{aligned}
\mathcal{L}^{(5)}_{\rm NLO} &\sim \bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi \\
\mathcal{L}^{(6)}_{\rm NLO} &\sim {\rm Tr}(D_\mu F_{\rho\sigma}D_\mu F_{\rho\sigma}) + {\rm Tr}(D_\mu F_{\mu\sigma}D_\rho F_{\rho\sigma}) \\
&\quad + \bar\psi D_\mu \not{D} D_\mu \gamma_\mu\psi + \dots + (\bar\psi\psi)^2 + (\bar\psi\gamma_\mu\psi)^2 + \dots \\
&\quad + \underbrace{{\rm Tr}(D_\mu F_{\mu\sigma}D_\mu F_{\mu\sigma}) + \bar\psi D^3_\mu\gamma_\mu\psi}_{\textbf{Lorentz violating}}
\end{aligned}$$

  - No "twisted Pauli term" (since factor of μ makes NNLO)
  - No flavor or parity breaking in four-fermion terms (requires factors of μ)

$\Rightarrow$ Aside from Lorentz violation, $\mathcal{L}^{(6)}_{\rm NLO}$ breaks no more symmetries than $\mathcal{L}^{(5)}_{\rm NLO}$, i.e. both break chiral symmetry

Why does maximal twist work?

- Why are physical quantities automatically $O(a)$ improved?
- At quark level, maximal twist implies:

$$\begin{aligned}\mathcal{L}_{\rm NLO}^{(4+5)} &= \bar\psi \not{D} \psi + \mu \bar\psi i \gamma_5 \tau_3 \psi + ac \bar\psi i \sigma_{\mu\nu} F_{\mu\nu} \psi \\ &= \bar\psi_{\rm phys} \not{D} \psi_{\rm phys} + \mu \bar\psi_{\rm phys} \psi_{\rm phys} + ac \bar\psi_{\rm phys} \gamma_5 \tau_3 \sigma_{\mu\nu} F_{\mu\nu} \psi_{\rm phys}\end{aligned}$$

$\Rightarrow$ $O(a)$ corrections necessarily violate parity and flavor

$\Rightarrow$ Physical (parity-flavor conserving) quantities corrected only at $O(a^2)$

▷ Caveat: corrections enhanced if $m_\pi^0 \to 0$

- [Frezzotti & Rossi] show this holds also for operator matrix elements (e.g. f_π)

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References

- S. R. Sharpe and R. L. . Singleton, "Spontaneous flavor and parity breaking with Wilson fermions," Phys. Rev. D **58**, 074501 (1998)
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tmχPT in the GSM regime: $\mathcal{L}^{(5)}$

- $\mathcal{L}^{(5)}_{\rm NLO}$ transforms like a mass term under $SU(2)_L \times SU(2)_R$:

$$a\mathcal{L}^{(5)}_{\rm NLO} \sim a\bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi = \bar\psi_L \widetilde{A} i\sigma_{\mu\nu}F_{\mu\nu}\psi_R + \bar\psi_R \widetilde{A}^\dagger i\sigma_{\mu\nu}F_{\mu\nu}\psi_A$$

  - $\widetilde{A}$ is a spurion like M: $\mathcal{L}^{(5)}$ invariant if $\widetilde{A} \to U_L \widetilde{A} U_R^\dagger$
  - Set $\widetilde{A} = a$ at the end
- Enumeration of terms identical to that for M, but LECs will differ
- At LO in the GSM regime [Sharpe & Singleton]

$$\mathcal{L}^{(2)}_\chi \quad = \quad \frac{f^2}{4}{\rm tr}(D_\mu\Sigma D_\mu\Sigma^\dagger) - \frac{f^2}{4}{\rm tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi) - \frac{f^2}{4}{\rm tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A})$$

  - $\hat{A} = 2W_0\widetilde{A}$ with W_0 a new LEC depending on gauge action

$$\frac{W_0}{B_0} \sim \frac{\langle\pi|\bar\psi\sigma_{\mu\nu}F_{\mu\nu}\psi|\pi\rangle}{\langle\pi|\bar\psi\psi|\pi\rangle} \sim \Lambda^2_{\rm QCD}$$

NLO contribution from $\mathcal{L}^{(5)}$

[Bär, Rupak & Shoresh]

$$
\begin{aligned}
\mathcal{L}_\chi^{(4)} \quad = \quad & -L_2 \mathrm{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger)\mathrm{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger) + L_{45}\mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \\
& + L_5 \left\{ \mathrm{tr}\left[(D_\mu \Sigma^\dagger D_\mu \Sigma)(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \right] - \mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi)/2 \right\} \\
& - L_{68} \left[\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \right]^2 - L_8 \left\{ \mathrm{tr}\left[(\chi^\dagger \Sigma + \Sigma^\dagger \chi)^2 \right] - \left[\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \right]^2 /2 \right\} \\
& - L_7 \left[\mathrm{tr}(\chi^\dagger \Sigma - \Sigma^\dagger \chi) \right]^2 + i L_{12} \mathrm{tr}(L_{\mu\nu} D_\mu \Sigma D_\nu \Sigma^\dagger + p.c.) + L_{13} \mathrm{tr}(L_{\mu\nu} \Sigma R_{\mu\nu} \Sigma) \\
& + W_{45} \mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) - W_{68} \mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi)\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \\
& - W'_{68} \left[\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \right]^2 + W_{10} \mathrm{tr}(D_\mu \hat{A}^\dagger D_\mu \Sigma + D_\mu \Sigma^\dagger D_\mu \hat{A})
\end{aligned}
$$

- Simplified using $SU(2)$ relations, dropped HECs
- New spurion can now be fixed: $\hat{A}, \hat{A}^\dagger \to 2W_0 a \equiv \hat{a}$
- Four new (dimensionless) LECs at NLO, expect $W_i \sim 1/(4\pi)^2$, but depend on gauge action
- Only three effect physical quantities (one linear combination is redundant)

What about $\mathcal{L}^{(6)}_{\rm NLO}$?

- Lorentz and chiral invariant terms give multiplicative a^2 corrections, which are of NNLO:

$$a^2 \mathrm{Tr}(D_\mu F_{\rho\sigma} D_\mu F_{\rho\sigma}) + \dots + a^2 \bar\psi D_\mu \not{D} D_\mu \gamma_\mu \psi + \dots \longrightarrow a^2 \mathrm{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger)$$

- Four-fermion operators violate chiral symmetry, but lead to no new $O(a^2)$ terms in $\mathcal{L}_\chi$ [Bär, Rupak & Shoresh]

$$(\bar\psi\psi)^2 + (\bar\psi\gamma_\mu\psi)^2 + \dots \longrightarrow \mathrm{tr}(\hat{A}^\dagger\Sigma + p.c.)^2$$

- Lorentz violating terms lead to Lorentz violating, chirally symmetric terms:

$$a^2 \mathrm{Tr}(D_\mu F_{\mu\sigma} D_\mu F_{\mu\sigma}) + a^2 \bar\psi D_\mu^3 \gamma_\mu \psi \longrightarrow a^2 \mathrm{tr}(D_\mu^2 \Sigma D_\mu^2 \Sigma^\dagger)$$

  but these are of NNNLO

- **CONCLUSION: $\mathcal{L}^{(6)}_{\rm NLO}$ leads to no new terms at NLO**

What if we NP improve action?

$$
\begin{aligned}
\mathcal{L}_\chi \quad = \quad & \frac{f^2}{4}\mathrm{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4}\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) - \frac{f^2}{4}\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \\
& - L_1 \mathrm{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger)^2 - L_2 \mathrm{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger)\mathrm{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger) \\
& + L_{45}\mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) - L_{68}\left[\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi)\right]^2 \\
& + W_{45}\mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) - W_{68}\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi)\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \\
& - W'_{68}\left[\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})\right]^2 + W_{10}\mathrm{tr}(D_\mu \hat{A}^\dagger D_\mu \Sigma + D_\mu \Sigma^\dagger D_\mu \hat{A})
\end{aligned}
$$

- Terms linear in A are removed
- Exception: W_{10}, which describes pionic matrix elements of A_μ and V_μ
  - Can set $W_{10} \to 0$ if NP improve axial current (vector current discretization errors are automatically improved)
- Term quadratic in A remains, though the value of W'_{68} will change

Summary

- Discretization errors in PGB masses, interactions and matrix elements involving V_μ, A_μ, s and p are described by a few additional constants throughout the "tm-plane"
- At LO one new constant (W_0), but will see unphysical
- At NLO have
  - 3 physical constants without improvement
  - 2 constants if NP $O(a)$ improve action
  - 1 constant if also improve axial current
- Thus expect predictions!

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  - Defining the twist angle
  - Results for $m_q \sim a^2\Lambda^3_{\rm QCD}$

Additional references for this section

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Implications of tmχPT: LO

$$\mathcal{L}_{\chi,\mathrm{LO}} = \frac{f^2}{4}\mathrm{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4}\mathrm{tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi) - \frac{f^2}{4}\mathrm{tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A})$$

- Takes same form as in continuum tmQCD if use $\chi' = \chi + \hat{A}$

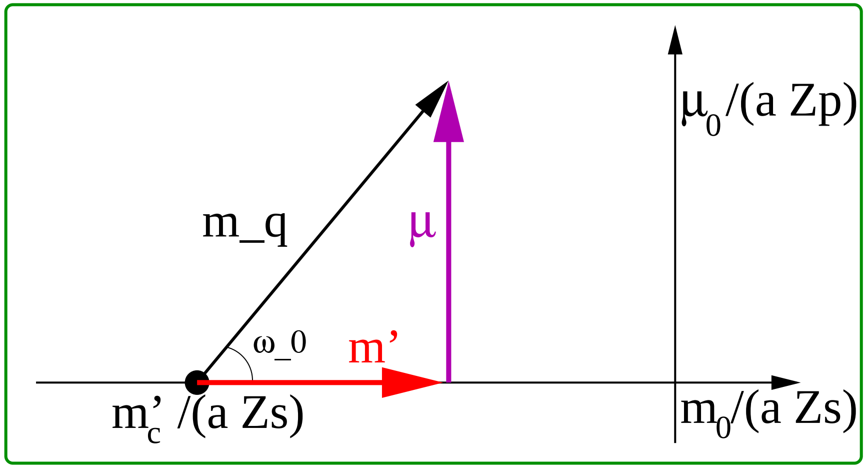
$$\mathcal{L}_{\chi,\mathrm{LO}} = \frac{f^2}{4}\mathrm{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4}\mathrm{tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi')$$

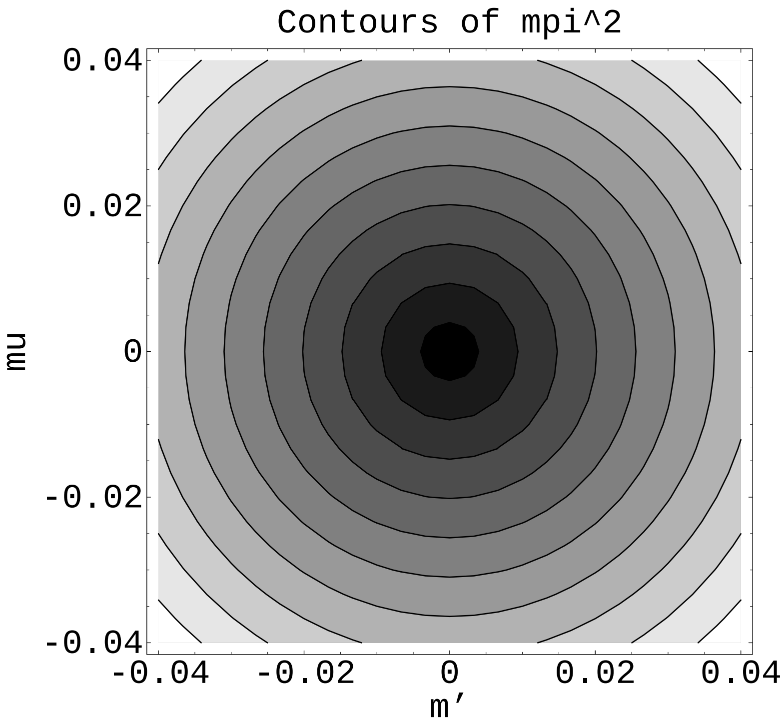
  - Corresponds to $O(a)$ shift in untwisted mass m and thus of $O(a^2)$ in m_c
    $m \to m' = m + a\frac{W_0}{B_0} = Z_S^{-1}\frac{m_0 - m_c}{a} + a\frac{W_0}{B_0}\,, \qquad \Delta m_c = -a^2\frac{Z_S W_0}{B_0}$
  - But m_c not known *a priori*
  - If determine $m_c' = m_c + \Delta m_c$ non-perturbatively, e.g. using $m_\pi^2 \propto m'$, then automatically include $O(a)$ shift

  $\Rightarrow$ W_0 is not a measurable parameter

- LO pion interactions have no $O(a)$ corrections for any twist angle!

tmχPT at LO: Summary





Condensate aligns with shifted mass, and physics independent of ω_0

tmχPT at NLO

Rewrite $\mathcal{L}_\chi$ in terms of χ' [Sharpe & Wu]

$$
\begin{aligned}
\mathcal{L}_\chi \quad = \quad & \frac{f^2}{4}\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger) - \frac{f^2}{4}\mathrm{tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi') \\
& -L_1\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger)^2 - L_2\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger) \\
& +L_{45}\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi') - L_{68}\left[\mathrm{tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi')\right]^2 \\
& +\widetilde{W}\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) - W\mathrm{tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi')\mathrm{tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) \\
& -W'\left[\mathrm{tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A})\right]^2
\end{aligned}
$$

- Shifted LECs (scale invariant):
  $\widetilde{W} = W_{45} - L_{45}\,, \quad W = W_{68} - 2L_{68}\,, \quad W' = W'_{68} - W_{68} + L_{68}$
- W, W' cause small misalignment of vacuum with χ'
- Skip details, and give examples of results

m_π at NLO in tmχPT

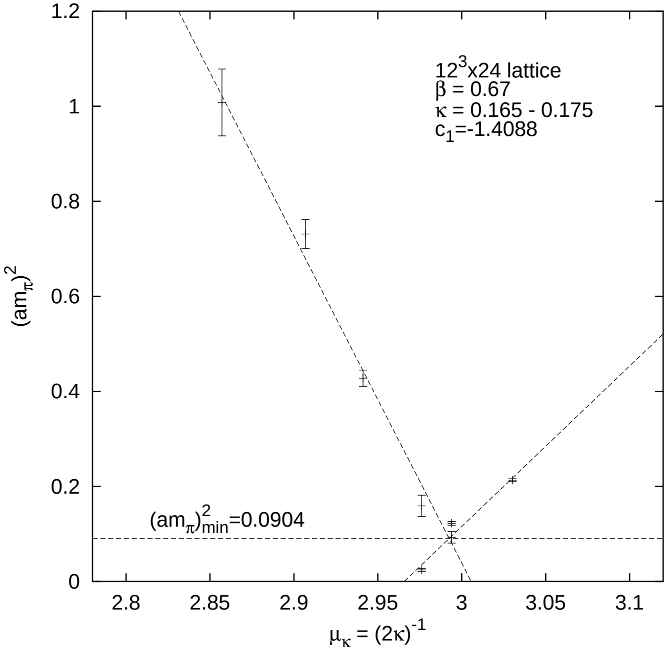
$$m^2_{\pi\pm} = |\chi'| + \textbf{cont. 1-loop chiral logs}$$
$$+\frac{16}{f^2}\left[|\chi'|^2(2L_{68}-L_{45}) + |\chi'|\hat{a}\cos\omega_0(2W-\widetilde{W}) + 2\hat{a}^2(\cos\omega_0)^2 W'\right]$$

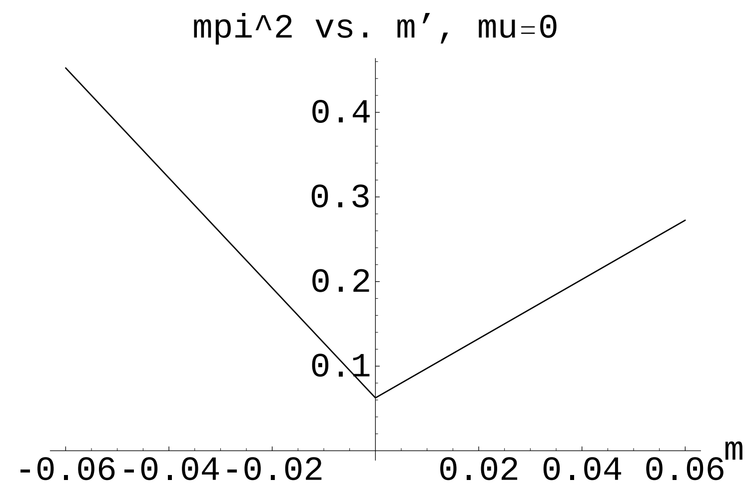
[Scorzato; Sharpe & Wu]

- Automatic $O(a)$ improvement at $\omega_0 = \pi/2 + O(a)$ [Frezzotti & Rossi]
- Absence of $O(a^2)$ term at $\omega_0 = \pi/2$ not generic
- Alternatively, Wilson-averaging ($\cos\omega_0 \leftrightarrow -\cos\omega_0$) cancels $O(a)$ term

tmχPT vs. lattice data on Wilson axis

$$m^2_{\pi\pm} = |\chi'| + \textbf{cont. 1-loop chiral logs}$$
$$+\frac{16}{f^2}\left[|\chi'|^2(2L_{68}-L_{45}) + |\chi'|\hat{a}\cos\omega_0(2W-\widetilde{W}) + 2\hat{a}^2(\cos\omega_0)^2 W'\right]$$



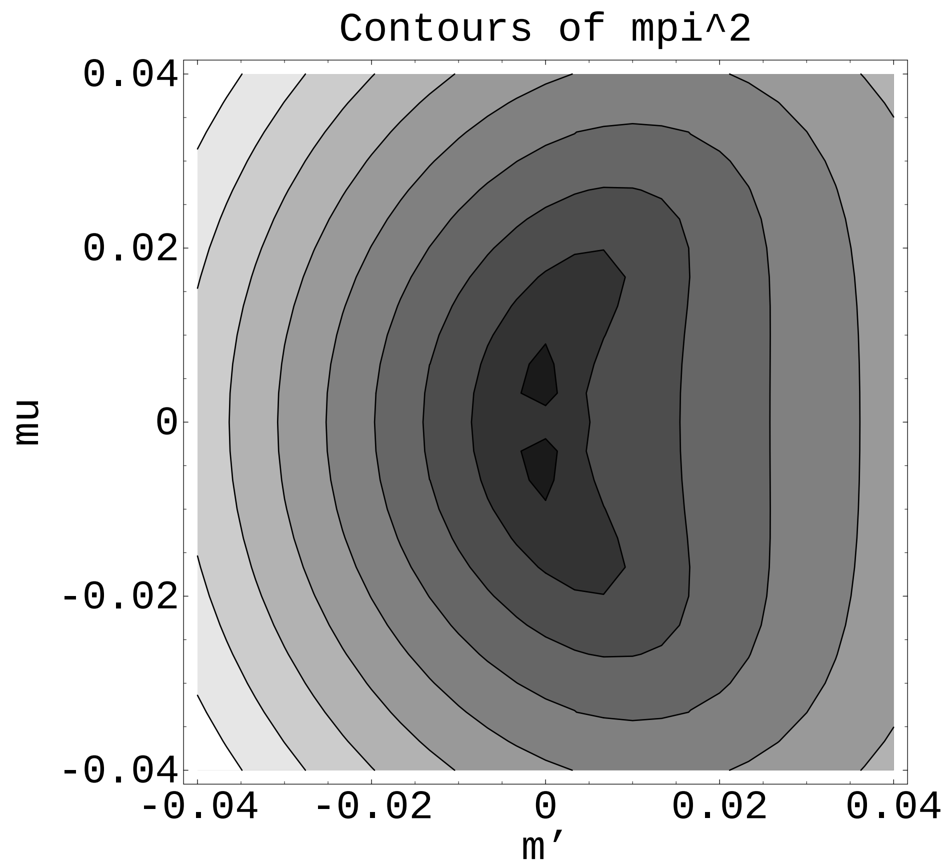


(Chiral logs set to zero)

[Farchioni *et al*,hep-lat/0410031]

- Clear antisymmetry of $\approx 30\% \sim a\Lambda^2$ with $\Lambda \approx 300$ MeV
- Non-vanishing minimum pion mass due to W'

ω_0 not redundant at NLO



- Contours of m_π^2 in twisted mass plane
  - LECs chosen to roughly fit data of [Farchioni04]

Predictions from tmχPT at NLO

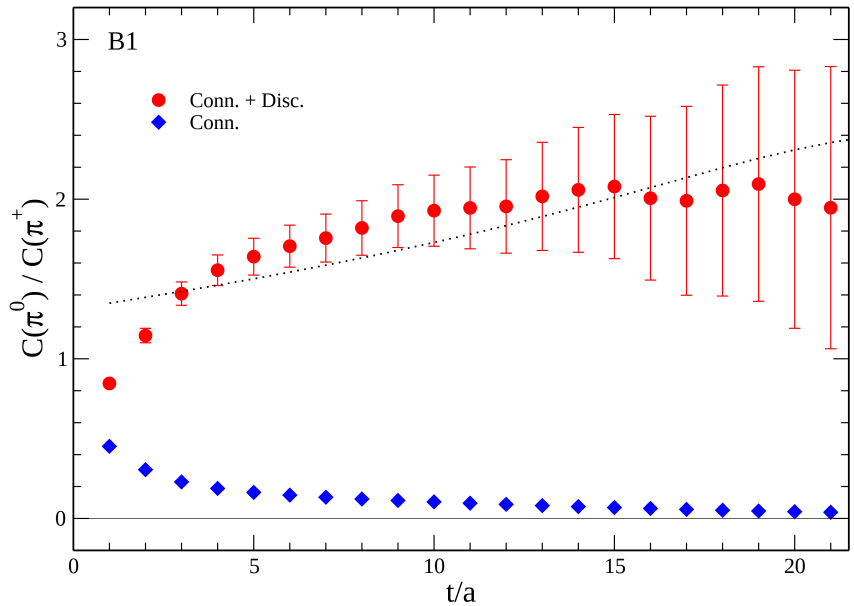
- $O(a)$ terms in $m_\pi^\pm$, $\langle 0|P^\mp|\pi^\pm\rangle$, $\langle\pi|S^0|\pi\rangle$, f_π, $m_{\rm PCAC}$, ...given in terms of two coefficients W and $\widetilde{W}$
  - Fits by [Aoki & Bar] (to quenched data) work reasonably
- Isospin splitting in pion multiplet:

$$\begin{aligned} m_{\pi^0}^2 - m_{\pi^\pm}^2 &= -\frac{32W'\hat{a}^2}{f^2}(\sin\omega_0)^2 + O(a^3) \\ &= -\frac{32W'\hat{a}^2}{f^2}\frac{\mu^2}{m'^2+\mu^2} + O(a^3) \end{aligned}$$

  - $O(a^2)$ for all ω_0, maximal at maximal twist
  - Calculated numerically (requires quark-disconnected contractions)
    - $\Delta m_\pi^2 \approx -(160\mathrm{MeV})^2$ with improved gauge action ($a \approx 0.09$ fm) [Boucaud, 0803.0224]
  - Coefficient W' also determines phase-structure in Aoki regime
  - Include flavor-breaking in loops? NNLO effect
    - Ignored in present chiral fits: not clear that this is justified
    - But crucial to include corresponding effects in SχPT fits

More on flavor-breaking

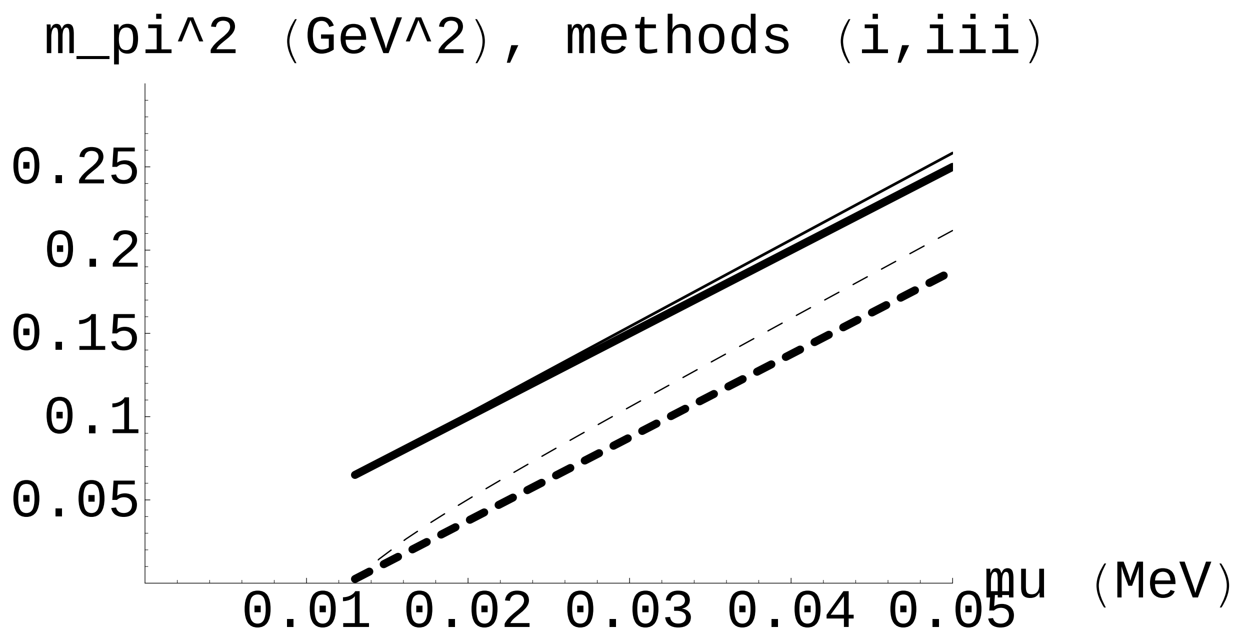
[Boucaud, 0803.0224]



- Disconnected contribution dominant!
- π^0 lighter

Yet more on flavor-breaking

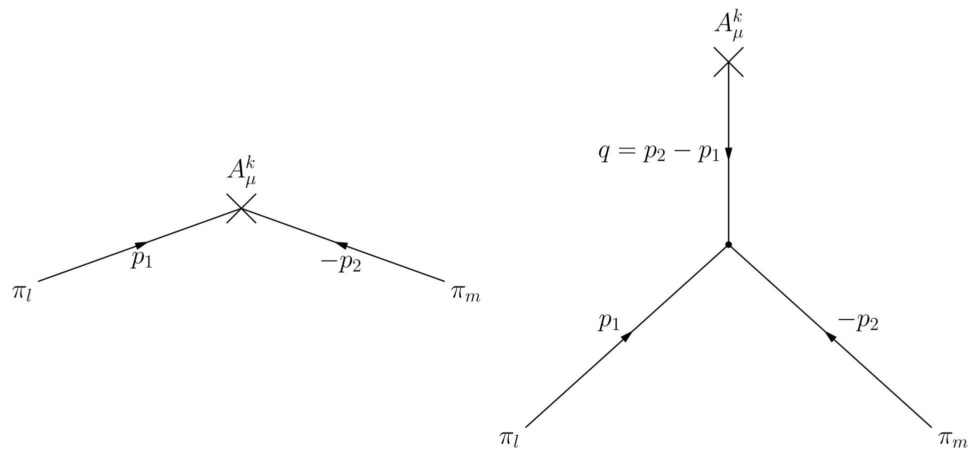
Charged (solid lines) and neutral (dashed) pion mass-squareds at maximal twist (values illustrative only; scale for μ should be GeV)



- Second-order endpoint when $m_{\pi^0} \to 0$
- Must avoid, since leads to unphysical effects
- Requires $m_{\pi^\pm} > 250\,\mathrm{MeV}$?

Lessons from tmχPT: parity-flavor breaking

- Automatic improvement at maximal twist only holds for physical quantities
- Unphysical quantities are $O(a)$, and provide another window on discretization errors
- e.g. axial and pseudoscalar form factors of pion:
  $\langle \pi_a | \hat{A}^a_\mu, \hat{P}^a | \pi_3 \rangle$
  $\langle \pi_a | \hat{A}^3_\mu, \hat{P}^3 | \pi_a \rangle$
  $\langle \pi_3 | \hat{A}^3_\mu, \hat{P}^3 | \pi_3 \rangle$



- Example of results:

$$\langle \pi_a(p_2) | \hat{P}^3 | \pi_a(p_1) \rangle = \frac{16\hat{a}\sin\omega_0 i B_0}{f^2}\left[+W - \widetilde{W} + \frac{2\hat{a}\cos\omega_0 W'}{q^2 + m^2_{\pi_3}} + \frac{(\widetilde{W}/2 - W)q^2}{q^2 + m^2_{\pi_3}} \right]$$

- Require quark-disconnected contractions $\Rightarrow$ not simple to calculate

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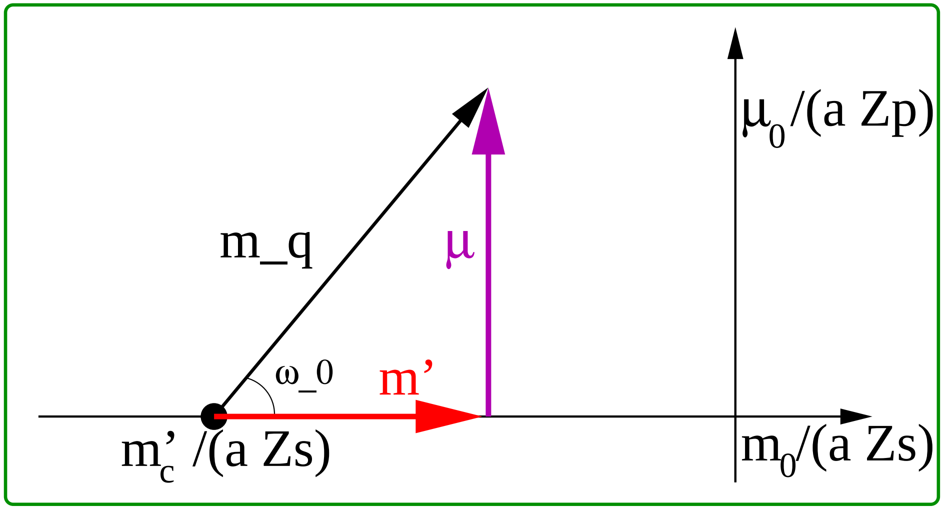
How determine m_c and the twist angle?



- Need $\omega_0 = \pi/2 + O(a)$, so if $\mu \sim a\Lambda^2_{\rm QCD}$, then need $m' \sim a^2\Lambda^3_{\rm QCD}$
- Traditional $m_\pi \to 0$ method fails at desired accuracy
  - Phase structure for $m_q \sim a^2\Lambda^3_{\rm QCD}$
    $\Rightarrow m_\pi$ does not vanish or vanishes over a range
- Now standard to determine m_c from $m_{\rm PCAC} = 0$ at smallest μ
  - Sufficiently accuracy for $\mu \gtrsim m_s/6$ [Boucaud, 0803.0224]
  - Earlier problems (non-smooth $\mu \to 0$ extrapolations—“bending”) resolved by improved accuracy in determination of m_c

tmχPT predictions for "PCAC method"

- Fix μ and scan in m until $m_{\rm PCAC}=0$, with

$$m_{PCAC} \equiv \frac{\langle \partial_\mu A^a_\mu(x) P^a(y)\rangle}{2\langle P^a(x) P^a(y)\rangle} \qquad (a=1,2)$$

- Equivalent to enforcing parity restoration in particular correlator:

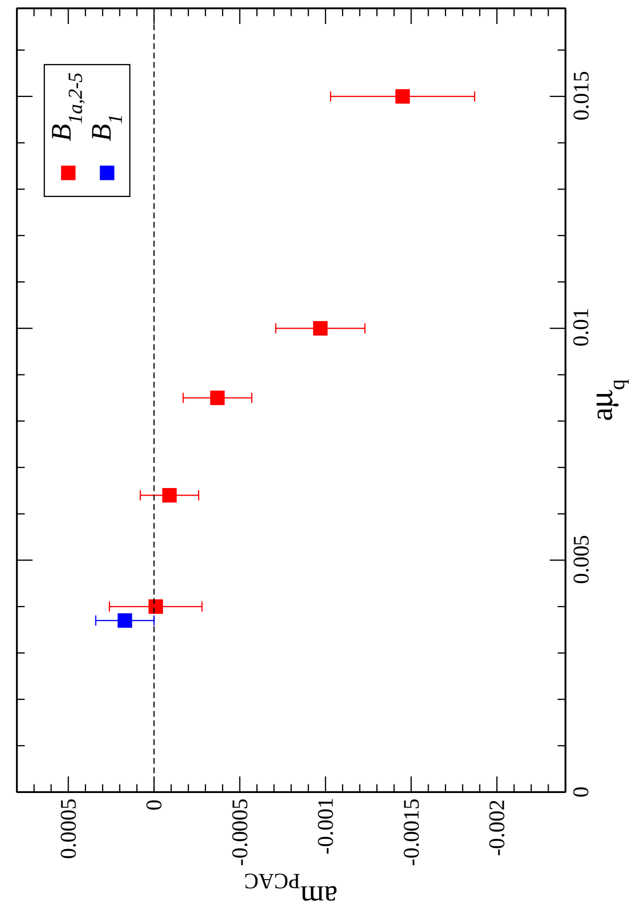
$$\langle A^1_\mu(x) P^1(y)\rangle = 0 \quad \Rightarrow \quad \langle V^{\rm phys,1}_\mu(x) P^{\rm phys,1}(y)\rangle = 0$$

- tmχPT implies that PCAC method gives

$$\omega_0 = \frac{\pi}{2} + \frac{16\hat{a}W}{f^2}$$

  - Numerically find correction term $\sim 10\%$ which is of expected size ($\sim a\Lambda_{\rm QCD}$) [Boucaud, 0803.0224]
  - Significantly reduced compared to (quenched) unimproved gauge action

Numerical results for “PCAC method”



[Boucaud, 0803.0224]

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Some simulations relevant for this section

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