Applications of Chiral Perturbation theory to lattice QCD (III)

First part adapted from [hep-lat/0607016] , second part new.

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Outline of Lecture 3

- Partial quenching and PQχPT
  - What is partial quenching?
  - Developing PQχPT
  - Results and outlook
- $m_u = 0$ and the validity of PQ theories

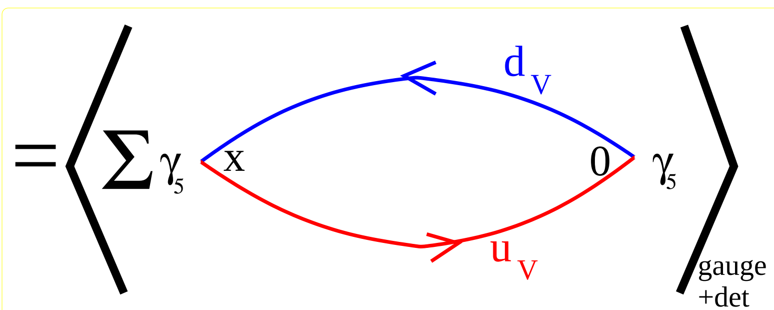
References for Partial Quenching

- A. Morel, *"Chiral logarithms in quenched QCD"* J.Phys. (Paris) **48**, 111 (1987)
- S. R. Sharpe, *"Chiral Logarithms In Quenched m_π and f_π,"* Phys. Rev. D **41**, 3233 (1990).
- C. W. Bernard and M. F. Golterman, *"Chiral perturbation theory for the quenched approximation of QCD,"* Phys. Rev. D **46**, 853 (1992)
- S. R. Sharpe, *"Quenched chiral logarithms,"* Phys. Rev. D **46**, 3146 (1992)
- C. W. Bernard and M. F. Golterman, *"Partially quenched gauge theories and an application to staggered fermions,"* Phys. Rev. D **49**, 486 (1994)
- P. H. Damgaard, *et al* "The microscopic spectral density of the QCD Dirac operator," Nucl. Phys. B **547**, 305 (1999)
- P. H. Damgaard and K. Splittorff, *"Partially quenched chiral perturbation theory and the replica method,"* Phys. Rev. D **62**, 054509 (2000)
- S. R. Sharpe and N. Shoresh, *"Physical results from unphysical simulations,"* Phys. Rev. D **62**, 094503 (2000)
- S. R. Sharpe and N. Shoresh, *"Partially quenched chiral perturbation theory without Φ_0,"* Phys. Rev. D **64**, 114510 (2001)
- S. R. Sharpe and R. S. Van de Water, "Unphysical operators in partially quenched QCD," Phys. Rev. D **69**, 054027 (2004)

What is Partially Quenched QCD?

- Explain with example of pion correlator:

$$
\begin{aligned}
C_\pi(\tau) &= -\left\langle \sum_{\vec{x}} \bar{u}\gamma_5 d(\vec{x},\tau)\; \bar{d}\gamma_5 u(0) \right\rangle \\
&\equiv -\frac{1}{Z}\int DU \prod_q Dq D\bar{q} e^{-S_{\rm gauge} - \int_x \sum_q \bar{q}(\not{D}+m_q)q} \sum_{\vec{x}} \bar{u}\gamma_5 d(\vec{x},\tau)\; \bar{d}\gamma_5 u(0) \\
&= \frac{1}{Z}\int DU \prod_q \det(\not{D}+m_q) e^{-S_{\rm gauge}} \sum_{\vec{x}} {\rm tr}\left[\gamma_5 \left(\frac{1}{\not{D}+m_d}\right)_{x0} \gamma_5 \left(\frac{1}{\not{D}+m_u}\right)_{0x}\right]
\end{aligned}
$$

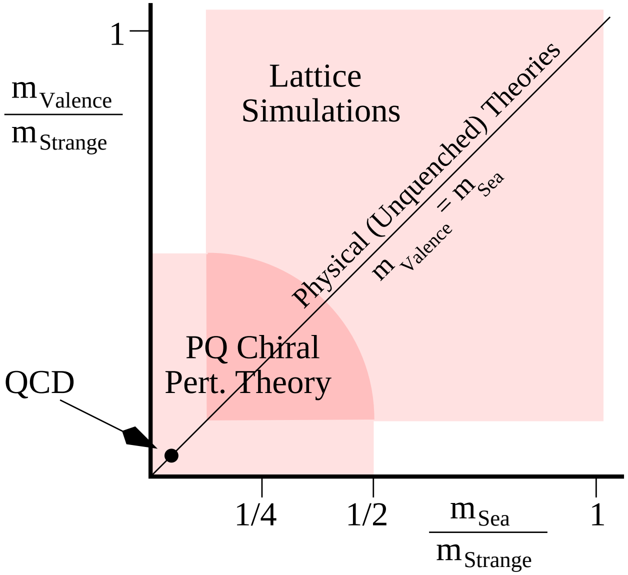


$\propto f_\pi^2 e^{-m_\pi \tau} + \text{exp. suppressed}$

- "sea" quarks in determinant; "valence" in propagators
- **Partial Quenching:** $m_{\rm val} \neq m_{\rm sea}$**—many different** $m_{\rm val}$ **for each** $m_{\rm sea}$
- **Numerically cheap—can we make use of this extra information?**

PQQCD needs PQχPT

- **Use PQQCD as a tool to learn about QCD, not as a model of QCD**
  - PQQCD is unphysical, e.g. not unitary
  - Intermediate and external "states" differ, e.g. $\pi_V\pi_V \longrightarrow \pi_S\pi_S \longrightarrow \pi_V\pi_V$



- **Need PQχPT in order to extrapolate to QCD**
  - ⇒ must be in the quark-mass regime where χPT is valid
  - Extends the range over which can match lattice and χPT

- **Subspace with $m_{\rm val} = m_{\rm sea}$ are physical QCD-like theories**
  - PQχPT must match χPT on subspace
  - LECs in PQχPT include those appearing in χPT, plus a few (sometimes none) additional unphysical ones

Historical comment on nomenclature

- Why called partially quenched? Why not partially unquenched?
- Bad old days: quenched approximation $m_{\rm sea} \to \infty$

  $\Rightarrow$ $\det(\not{D} + m_q) \to \text{constant}$

  $\Rightarrow$ **No quark loops**

  $\Rightarrow$ $Z_{\rm QCD} \to Z_{\rm QQCD} = \int DU e^{-S_{\rm gauge}} = Z_{\rm gauge}$
- Unphysical nature of quenched QCD shows up various ways, e.g. $\langle \bar\psi\psi \rangle \to \infty$ as $m_{\rm val} \to 0$
- Partial quenching is in one sense a less extreme version of quenching, and thus the name
- If $m_{\rm sea} \gg \Lambda_{\rm QCD}$ then PQQCD, like quenched QCD, only qualitatively related to QCD
- **Consider here only the case when $m_{\rm sea} \ll \Lambda_{\rm QCD}$ so one can use χPT and relate PQCD to QCD quantitatively**

Morel's formulation of (P)QQCD

- **IDEA:** commuting spin-$\frac{1}{2}$ fields (ghosts) $\widetilde{q}$ give determinant which cancels that from valence quarks

$$\int D\bar{q}Dq\, e^{-\bar{q}(\not{D}+m_q)q} = \det(\not{D}+m_q)$$

$$\int D\widetilde{q}^\dagger D\widetilde{q}\, e^{-\widetilde{q}^\dagger(\not{D}+m_q)\widetilde{q}} = \frac{1}{\det(\not{D}+m_q)}$$

- To formulate PQQCD need three types of "quark"
  - valence quarks $q_{V1}, q_{V2}, \dots q_{VN_V}$ $(N_V = 2, 3, \dots)$
  - sea quarks $q_{S1}, q_{S2}, \dots q_{SN}$ $(N = 2, 3)$
  - ghosts $\widetilde{q}_{V1}, \widetilde{q}_{V2}, \dots \widetilde{q}_{VN_V}$ $(N_V = 2, 3, \dots)$
- Ghosts are degenerate with corresponding valence quarks

Morel's formulation (cont.)

Partition function reproduces that which is simulated:

$$Z_{\rm PQ} = \int DU e^{-S_{\rm gauge}} \int \prod_{i=1}^{N_V} \left(D\bar{q}_{V\,i} Dq_{V\,i} D\widetilde{q}^{\dagger}_{V\,i} D\widetilde{q}_{V\,i} \right) \prod_{j=1}^{N} \left(D\bar{q}_{S\,j} Dq_{S\,j} \right) \times$$

$$\times \exp\left[-\sum_{i=1}^{N_V} \bar{q}_{V\,i} (\not{D} + m_{V\,i}) q_{V\,i} - \sum_{j=1}^{N} \bar{q}_{S\,j} (\not{D} + m_{S\,j}) q_{S\,j} - \sum_{k=1}^{N_V} \widetilde{q}^{\dagger}_{V\,k} (\not{D} + m_{V\,k}) \widetilde{q}_{V\,k} \right]$$

$$= \int DU e^{-S_{\rm gauge}} \prod_{i=1}^{N_V} \left(\frac{\det(\not{D} + m_{V\,i})}{\det(\not{D} + m_{V\,i})} \right) \prod_{j=1}^{N} \det(\not{D} + m_{S\,j})$$

$$= \int DU e^{-S_{\rm gauge}} \prod_{j=1}^{N} \det(\not{D} + m_{S\,j})$$

$$= Z_{\rm QCD-like}$$

Compact Notation

- Collect all fields into $(N+2N_V)$-dim vectors:

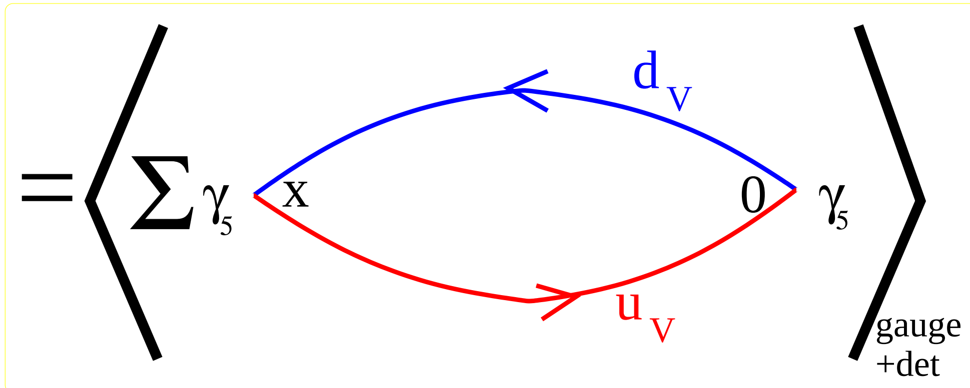
$$
\begin{aligned}
Q &= \Big(\underbrace{q_{V1}, q_{V2}, \ldots, q_{VN_V}}_{\text{valence}}, \underbrace{q_{S1}, q_{S2}, \ldots, q_{SN}}_{\text{sea}}, \underbrace{\widetilde{q}_{V1}, \widetilde{q}_{V2}, \ldots, \widetilde{q}_{VN_V}}_{\text{ghost}}\Big) \\
\overline{Q}^{tr} &= \Big(\underbrace{\bar{q}_{V1}, \bar{q}_{V2}, \ldots, \bar{q}_{VN_V}}_{\text{valence}}, \underbrace{\bar{q}_{S1}, \bar{q}_{S2}, \ldots, \bar{q}_{SN}}_{\text{sea}}, \underbrace{\widetilde{q}^{\dagger}_{V1}, \widetilde{q}^{\dagger}_{V2}, \ldots, \widetilde{q}^{\dagger}_{VN_V}}_{\text{ghost}}\Big) \\
\mathcal{M} &= \Big(\underbrace{m_{V1}, m_{V2}, \ldots, m_{VN_V}}_{\text{valence}}, \underbrace{m_{S1}, m_{S2}, \ldots, m_{SN}}_{\text{sea}}, \underbrace{m_{V1}, m_{V2}, \ldots, m_{VN_V}}_{\text{ghost=valence}}\Big)
\end{aligned}
$$

- Then can write action and partition function as:

$$
\begin{aligned}
S_{\mathrm{PQ}} &= S_{\mathrm{gauge}} + \overline{Q}(\not{D} + \mathcal{M})Q \\
Z_{\mathrm{PQ}} &= \int DU\, D\bar{Q} DQ\; e^{-S_{\mathrm{PQ}}}
\end{aligned}
$$

Formal representation of PQ correlator

$C_\pi^{\rm PQ}(\tau)$



$$= \quad Z_{\rm PQ}^{-1} \int DU \prod_{j=1}^{N} \det(\not{D} + m_{Sj}) e^{-S_{\rm gauge}}$$

$$\times \sum_{\vec{x}} {\rm tr}\left[\gamma_5 \left(\frac{1}{\not{D} + m_{Vd}}\right)_{x0} \gamma_5 \left(\frac{1}{\not{D} + m_{Vu}}\right)_{0x}\right]$$

$$= \quad Z_{\rm PQ}^{-1} \int DU D\overline{Q} DQ \; e^{-S_{\rm PQ}} \sum_{\vec{x}} \bar{u}_V \gamma_5 d_V (\vec{x}, \tau) \; \bar{d}_V \gamma_5 u_V (0)$$

$$Q \quad = \quad \Big(\underbrace{q_{V1}, q_{V2}, \ldots, q_{VN_V}}_{\rm valence}, \underbrace{q_{S1}, q_{S2}, \ldots, q_{SN}}_{\rm sea}, \underbrace{\widetilde{q}_{V1}, \widetilde{q}_{V2}, \ldots, \widetilde{q}_{VN_V}}_{\rm ghost} \Big)$$

What have we learned about PQQCD?

- Well defined statistical system describing correlators in Euclidean space
  - Can use to represent individual contractions in complicated processes, e.g. $\pi\pi \longrightarrow \pi\pi$
- Regained unitarity, but at the cost of introducing ghosts
- Shows the ways the PQ theory is unphysical
  - violates spin-statistics theorem
  - loses causality and positivity in Minkowski space
  - loses reflection positivity in Euclidean space
- Unphysical nature shows up in various ways:
  - Double poles in correlation functions
  - Correlators involving multi-particle states do not have exponential fall-off in time, and have contributions which diverge in infinite volume $\Rightarrow$ cannot define scattering amplitudes [Lin et al]
- **Can we develop an EFT describing PQQCD including its unphysical nature?**

Key property of PQQCD

- "Anchored" to physical QCD-like theories
- If $m_{Vu} = m_{Sj}$ and $m_{Vd} = m_{Sk}$ then valence correlator *is* physical:

$$\begin{aligned}
C_\pi^{\rm PQ}(\tau) &= Z_{\rm PQ}^{-1} \int DU\, D\overline{Q} DQ\; e^{-S_{\rm PQ}} \sum_{\vec x} \bar u_V \gamma_5 d_V(\vec x,\tau)\; \bar d_V \gamma_5 u_V(0) \\
&= Z_{\rm PQ}^{-1} \int DU\, D\overline{Q} DQ\; e^{-S_{\rm PQ}} \sum_{\vec x} \bar q_{Sj} \gamma_5 q_{Sk}(\vec x,\tau)\; \bar q_{Sk} \gamma_5 q_{Sj}(0) \\
&= Z_{\rm QCD-like}^{-1} \int DU \prod_{i=1}^{N} D\bar q_{Si} Dq_{Si}\; e^{-S_{\rm QCD-like}} \\
&\qquad \times \sum_{\vec x} \bar q_{Sj} \gamma_5 q_{Sk}(\vec x,\tau)\; \bar q_{Sk} \gamma_5 q_{Sj}(0) \\
&= C_\pi^{\rm QCD-like}(\tau)
\end{aligned}$$

- Example of enhanced (V $\leftrightarrow$ S) symmetry in PQ theory

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Methods for developing PQχPT

- "Supersymmetric" method based on Morel's formulation [Bernard & Golterman]
- "Replica" method adjusting loop contributions by adjusting $N_{\rm sea}$ [Damgaard & Splittorf]
  - Formalizes "Quark-line" method accounting by hand for quarks in loops [Sharpe]
- Give same results to date—likely equivalent
- Use supersymmetric method here (with addition of some quark-line method when considering staggered fermions)

Symmetries of PQQCD

$$Q = \Big(\underbrace{q_{V1}, q_{V2}, \ldots, q_{VN_V}}_{\text{valence}}, \underbrace{q_{S1}, q_{S2}, \ldots, q_{SN}}_{\text{sea}}, \underbrace{\widetilde{q}_{V1}, \widetilde{q}_{V2}, \ldots, \widetilde{q}_{VN_V}}_{\text{ghost}} \Big)$$

- Action of PQQCD looks like QCD

$$S_{\text{PQQCD}} = S_{\text{gauge}} + \overline{Q}(\not{D} + \mathcal{M})Q$$

- Naively, when $M \to 0$ have graded version of QCD chiral symmetry:

$$Q_{L,R} \longrightarrow U_{L,R} Q_{L,R}\,, \qquad \overline{Q}_{L,R} \longrightarrow \overline{Q}_{L,R} U^\dagger_{L,R} \qquad U_{L,R} \in SU(N_V + N|N_V)$$

- Apparent symmetry is $SU(N_V + N|N_V)_L \times SU(N_V + N|N_V)_R \times U(1)_V$
- In fact, there are subtleties in the ghost sector, but can ignore in perturbative calculations [Sharpe & Shoresh]

Brief primer on graded Lie groups

- U is graded: contains both commuting and anticommuting elements:

$$U = \begin{pmatrix} A & B \\ \underbrace{C}_{N_V+N} & \underbrace{D}_{N_V} \end{pmatrix}, \quad A, D \text{ commuting, } B, C \text{ anticommuting}$$

- If $U \in U(N_V + N|N_V)$ (fundamental representation) then

$$UU^\dagger = U^\dagger U = 1\,, \quad [\text{with } (\eta_1\eta_2)^* \equiv \eta_2^*\eta_1^*]$$

- Supertrace maintains cyclicity:

$$\text{str}U \equiv \text{tr}A - \text{tr}D \quad \Rightarrow \quad \text{str}(U_1U_2) = \text{str}(U_2U_1)$$

- For $U \in SU(N_V + N|N_V)$, superdeterminant is unity:

$$\text{sdet}U \equiv \exp[\text{str}(\ln U)] = \frac{\det(A - BD^{-1}C)}{\det(D)} \quad \Rightarrow \quad \text{sdet}(U_1U_2) = \text{sdet}U_1\text{sdet}U_2$$

Examples of $SU(N_V+N|N)$ matrices

$$U = \begin{pmatrix} SU(N_V+N) & 0 \\ 0 & SU(N_V) \end{pmatrix} \quad \Rightarrow \quad \mathrm{sdet} U = 1$$

$$U = \begin{pmatrix} e^{i\theta N_V} & 0 \\ 0 & e^{i\theta(N+N_V)} \end{pmatrix} \quad \Rightarrow \quad \mathrm{sdet} U = \frac{(e^{i\theta N_V})^{N+N_V}}{(e^{i\theta(N+N_V)})^{N_V}} = 1$$

- An overall phase rotation is not in $SU(N_V+N|N)$

$$U = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \quad \Rightarrow \quad \mathrm{sdet} U = \frac{e^{i\theta(N+N_V)}}{e^{i\theta N_V}} = e^{i\theta N}$$

- Thus $U(N_V+N|N_V) = [SU(N_V+N|N_V) \otimes U(1)]/Z_N$
- Group structure different if $N=0$ (quenched theory)

Follow same steps as for QCD

- Expand about $\mathcal{M} = 0$
  - *A posteriori* find that must take chiral limit with m_V and m_S in fixed ratio
  - Divergences if $m_V \to 0$ at fixed m_S [Sharpe]
- Graded chiral symmetry is broken by condensate
  - Have Goldstone bosons and fermions (but both spin 0)
- Develop low-energy EFT based on symmetries and symmetry breaking
  - Weaker theoretical basis than usual χPT since underlying theory is unphysical
  - PQχPT matches unphysical features of PQQCD (e.g. double poles)
- Most LECs in PQχPT are the same as those in χPT because QCD is a subset of PQQCD
  - Use PQQCD to determine physical parameters of QCD (and/or to improve chiral extrapolations)

Symmetry breaking in PQQCD

- Symmetry group ($M \to 0$): $\mathcal{G} = SU(N_V + N|N_V)_L \times SU(N_V + N|N_V)_R$
- For $\mathcal{M}$ diagonal, real and positive [Vafa & Witten] implies graded vector symmetry not spontaneously broken
  - ▷ Quark and ghost condensates equal if $m_V = m_S \to 0$
    $\langle q_V \bar{q}_V \rangle = \langle \tilde{q}_V \bar{\tilde{q}}_V \rangle = \langle q_S \bar{q}_S \rangle = \omega$
- Spontaneous chiral symmetry breaking in QCD $\Rightarrow \omega \neq 0$
  - $\Rightarrow$ We know pattern of symmetry breaking. Introducing order parameter

$$\Omega_{ij} = \langle Q_{L,i,\alpha,c} \overline{Q}_{R,j,\alpha,c} \rangle_{\rm PQ} \xrightarrow[\mathcal{G}]{} U_L \, \Omega \, U_R^\dagger$$

    we know $\Omega = \omega \times 1$ with standard masses $\Rightarrow$ vacuum manifold is $SU(N_V + N|N_V)$
  - ▷ Symmetry breaking is $\mathcal{G} \to \mathcal{H} = SU(N_V + N|N_V)_V$
- Can derive Goldstone's theorem using Ward identities for two-point Euclidean correlators
  - ▷ $(N + 2N_V)^2 - 1$ Goldstone "particles" created by operators $\overline{Q}\gamma_\mu\gamma_5 T^a Q$ with T^a a traceless generator of $SU(N_V + N|N_V)$

Moving to EFT

- In QCD, proceed as follows:
  - Having established GB poles in two-point functions, we know that they will also be present in higher-order correlation functions, and in cuts
  - χPT reproduces this behavior, while incorporating the chiral Ward identities, and yielding physical S-matrix
- In PQQCD, situation is worse:
  - Have GB poles in two-point functions
  - Have Ward identities between correlation functions
  - No Hamiltonian so cannot show that same poles appear in higher-order correlation functions, or in cuts (no complete sets of states)
  - In fact, can show that there are double poles (but no higher) in neutral correlators [Sharpe & Shoresh]
  - Cannot rely on Weinberg's argument to determine EFT since no S-matrix
  - Only "anchor" is fact that know EFT for QCD-like subspace
- For PQQCD must simply assume minimal change from QCD: assume that have **local** $\mathcal{L}_{\rm eff}$, constrained by symmetries
  - Saturates Ward identities and reproduces double poles

Constructing $\mathcal{L}_{\rm PQ}$: choice of Σ

- Follow method used for QCD:

$$\Omega/\omega \to \Sigma(x) \in SU(N_V+N|N)\,, \qquad \Sigma \underset{\mathcal{G}}{\longrightarrow} U_L\,\Sigma\,U_R^\dagger$$

- For standard masses, $\langle\Sigma\rangle = 1$, so define Goldstones by

$$\Sigma = \exp\left[\frac{2i}{f}\Phi(x)\right]\,, \qquad \Phi(x) = \begin{pmatrix} \phi(x) & \eta_1(x) \\ \eta_2(x) & \widetilde{\phi}(x) \end{pmatrix}$$

  - $\mathrm{sdet}\Sigma = 1 \Rightarrow \mathrm{str}\Phi = \mathrm{tr}\phi - \mathrm{tr}\widetilde{\phi} = 0$

- QCD GBs contained in Φ

$$\Phi(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \pi(x) & 0 \\ \underbrace{0}_{N_V} & \underbrace{0}_{N} & \underbrace{0}_{N_V} \end{pmatrix} \quad \Rightarrow \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Sigma_{\rm QCD} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Building blocks for PQχPT as for χPT, e.g.

$$L_\mu = \Sigma D_\mu \Sigma^\dagger \to U_L L_\mu U_L^\dagger\,, \qquad \mathrm{str}(L_\mu) = 0$$

- Power counting as in χPT

PQ chiral Lagrangian [Bernard & Golterman]

$$\mathcal{L}^{(2)} = \frac{f^2}{4}\mathrm{str}\left(D_\mu \Sigma D_\mu \Sigma^\dagger\right) - \frac{f^2}{4}\mathrm{str}(\chi\Sigma^\dagger + \Sigma\chi^\dagger)$$

$$\begin{aligned}\mathcal{L}^{(4)} = &-L_1\,\mathrm{str}(D_\mu\Sigma D_\mu\Sigma^\dagger)^2 - L_2\,\mathrm{str}(D_\mu\Sigma D_\nu\Sigma^\dagger)\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)\\ &+L_3\,\mathrm{str}(D_\mu\Sigma D_\mu\Sigma^\dagger D_\nu\Sigma D_\nu\Sigma^\dagger)\\ &+L_4\,\mathrm{str}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{str}(\chi^\dagger\Sigma + \Sigma^\dagger\chi) + L_5\,\mathrm{str}(D_\mu\Sigma^\dagger D_\mu\Sigma)[\chi^\dagger\Sigma + \Sigma^\dagger\chi])\\ &-L_6\left[\mathrm{str}(\chi^\dagger\Sigma + \Sigma^\dagger\chi)\right]^2 - L_7\left[\mathrm{str}(\chi^\dagger\Sigma - \Sigma^\dagger\chi)\right]^2 - L_8\,\mathrm{str}(\chi^\dagger\Sigma\chi^\dagger\Sigma + \mathrm{p.c.})\\ &+L_9\,i\mathrm{str}(L_{\mu\nu}D_\mu\Sigma D_\nu\Sigma^\dagger + p.c.) + L_{10}\,\mathrm{str}(L_{\mu\nu}\Sigma R_{\mu\nu}\Sigma^\dagger)\\ &+H_1\,\mathrm{str}(L_{\mu\nu}L_{\mu\nu} + p.c.) + H_2\,\mathrm{str}(\chi^\dagger\chi) + \mathrm{WZW}_{\mathrm{PQ}}\\ &+L_{\mathrm{PQ}}\mathcal{O}_{PQ}\end{aligned}$$

- $\chi = 2B_0\mathcal{M}$
- Same form as for QCD with $\mathrm{tr} \to \mathrm{str}$ plus one extra term ($\mathcal{O}_{\mathrm{PQ}}$)
- **How do the LECs related to those of QCD?**

Relating PQχPT to χPT

- If choose Σ to lie in QCD subspace

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Sigma_{\rm QCD} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  and sources do not connect subspaces, then

$$\mathcal{L}^{(2,4,\dots)}_{\rm PQ\chi PT}(\Sigma) \to \mathcal{L}^{(2,4,\dots)}_{\chi \rm PT}(\Sigma_{\rm QCD})$$

- If external fields in correlation function are from sea sector, then can show that all valence and ghost contributions cancel in intermediate states
  - $\Rightarrow$ Σ takes the form given above
  - PQχPT calculation collapses to one in χPT
- **Thus LECs in PQχPT are equal to those in χPT**
  - Results in the chiral regime from PQQCD give information about physical LECs

What about $\mathcal{O}_{\rm PQ}$?

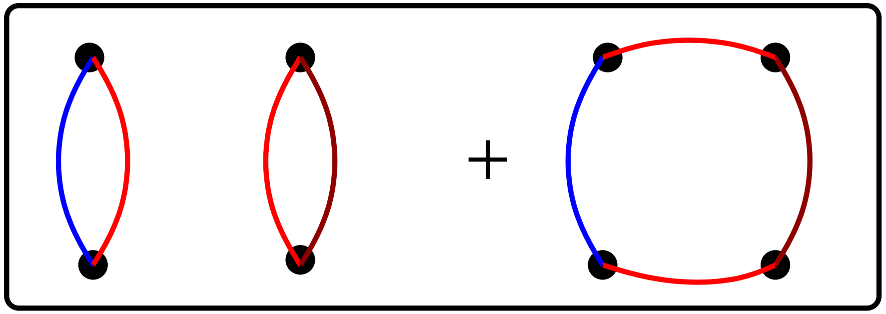
- Starting at NLO, at each order there are an increasing number of PQ operators that vanish on QCD subspace
- At NLO, only one such operator [Sharpe & Van de Water]

$$\begin{aligned}\mathcal{O}_{\rm PQ} &= {\rm str}(D_\mu\Sigma D_\nu\Sigma^\dagger D_\mu\Sigma D_\nu\Sigma^\dagger) \\ &\quad - \frac{1}{2}{\rm str}(D_\mu\Sigma D_\mu\Sigma^\dagger)^2 - {\rm str}(D_\mu\Sigma D_\nu\Sigma^\dagger){\rm str}(D_\mu\Sigma D_\nu\Sigma^\dagger) \\ &\quad + 2{\rm str}(D_\mu\Sigma D_\nu\Sigma^\dagger D_\mu\Sigma D_\nu\Sigma^\dagger)\end{aligned}$$

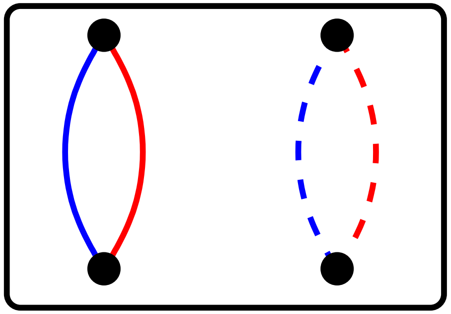
- Vanishes if $\Sigma \to \Sigma_{\rm QCD}$ due to Cayley-Hamilton relations for 3×3 matrices
- Does *not vanish* for general $\Sigma_{\rm PQ}$
- Appears in $\mathcal{L}^{(4)}_{\rm PQ\chi}$ with additional LEC
- Same is true for standard χPT if $N \geq 4$
- $\mathcal{O}_{\rm PQ}$ contributes to $\pi\pi$ scattering at NLO, but to m_π and f_π only at NNLO

Why is $\mathcal{O}_{\rm PQ}$ present?

- Because PQQCD allows isolation of individual Wick contractions, unlike QCD
- For example, $\pi^+ K^0$ scattering in QCD has two contractions



- Can separate these contractions in PQQCD, e.g.



- $\mathcal{O}_{\rm PQ}$ contributes to the PQQCD process, but not that in QCD
- Shows how PQQCD differs from QCD even if $m_V = m_S$

Calculating in PQχPT

- PQ Lagrangian at LO:

$$\mathcal{L}^{(2)} = \frac{f^2}{4}\mathrm{str}\left(D_\mu \Sigma D_\mu \Sigma^\dagger\right) - \frac{f^2}{4}\mathrm{str}(\chi\Sigma^\dagger + \Sigma\chi^\dagger)$$

- Insert expansion in Goldstone fields:

$$\Sigma = \exp\left[\frac{2i}{f}\Phi(x)\right], \qquad \Phi(x) = \begin{pmatrix} \phi(x) & \eta_1(x) \\ \eta_2(x) & \widetilde{\phi}(x) \end{pmatrix}, \qquad \mathrm{str}\Phi = 0$$

$$\begin{aligned}
\mathcal{L}^{(2)} &= \mathrm{str}(\partial_\mu\Phi\partial_\mu\Phi) + \mathrm{str}(\chi\Phi^2) + \ldots \\
&= \mathrm{tr}(\partial_\mu\phi\partial_\mu\phi + \partial_\mu\eta_1\partial_\mu\eta_2 - \partial_\mu\eta_2\partial_\mu\eta_1 - \partial_\mu\widetilde{\phi}\partial_\mu\widetilde{\phi}) \\
&\quad + \mathrm{tr}\left[(\phi^2 + \eta_1\eta_2)\begin{pmatrix} m_V & 0 \\ 0 & m_S \end{pmatrix}\right] - \mathrm{tr}(\widetilde{\phi}^2 m_V) - \mathrm{tr}(\eta_2\eta_1 m_V)
\end{aligned}$$

- ϕ part is like in QCD, except includes both valence and sea quarks
  - Propagator for "charged" meson $\bar{q}_1 q_2$ (either valence of sea) is $1/(p^2 + m_{12}^2)$, $m_{12}^2 = (\chi_1 + \chi_2)/2$

LO calculation (cont.)

$$\mathcal{L}^{(2)} = \mathrm{tr}(\partial_\mu \phi \partial_\mu \phi + \partial_\mu \eta_1 \partial_\mu \eta_2 - \partial_\mu \eta_2 \partial_\mu \eta_1 - \partial_\mu \widetilde{\phi} \partial_\mu \widetilde{\phi}) + \mathrm{tr}\left[(\phi^2 + \eta_1 \eta_2) \begin{pmatrix} m_V & 0 \\ 0 & m_S \end{pmatrix}\right] - \mathrm{tr}(\widetilde{\phi}^2 m_V) - \mathrm{tr}(\eta_2 \eta_1 m_V)$$

- $\widetilde{\phi}$ terms have wrong signs
  - Naively, propagator for "charged" ghost mesons $\bar{\widetilde{q}}_1 \widetilde{q}_2$ is $-1/(p^2 + m_{12}^2)$, $m_{12}^2 = (\chi_1 + \chi_2)/2$
  - But potential not minimized and functional integral not convergent!
  - More careful treatment of symmetries of PQQCD, maintaining convergence of ghost functional integral, concludes that naive result is OK in perturbation theory (but not non-perturbatively, e.g. in ϵ-regime, where should change $\widetilde{\phi} \to i\widetilde{\phi}$, $\Sigma^\dagger \to \Sigma^{-1}$) [Sharpe & Shoresh]
- Goldstone fermion propagators can have either sign (no convergence problems); actual signs important for cancellations

What about Φ_0?

- How implement $\mathrm{str}(\Phi) = \mathrm{tr}(\phi) - \mathrm{tr}(\widetilde{\phi}) = 0$?
  1. Use a basis of generators which is straceless:

     $\Phi = \sum_a \Phi_a T^a$ with $\mathrm{str}(T^a) = 0$
     - Analagous to not including the η' in QCD χPT
     - Clumsy in practice and not used
  2. Include identity component but then "integrate out"

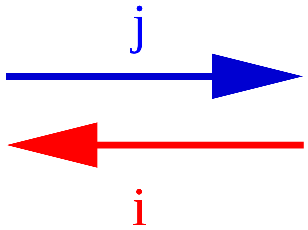
     $\Phi \to \Phi + \Phi_0/\sqrt{N}$ so that $\mathrm{str}\Phi = \sqrt{N}\Phi_0$

     $\mathcal{L}_{\mathrm{PQ}\chi} \to \mathcal{L}_{\mathrm{PQ}\chi} + m_0^2 \mathrm{str}(\Phi)^2/N$
     - Calculate propagators, then send $m_0^2 \to \infty$ within them
     - To make formally correct, must regularize with a cut-off (e.g. lattice) so that $(\partial_\mu \Phi_0)^2 < m_0^2 \Phi_0^2$ (trivial decoupling)
     - Really just a trick to implement stracelessness
     - Method used in practice
- Introducing Φ_0 has advantage of allowing use of "quark line" basis: $\Phi_{ij} \sim Q_i \overline{Q}_j$ for all i, j

Quark lines and double poles

- "Charged" particle propagators are simple:

$$\langle \Phi_{ij}\Phi_{ji}\rangle = \pm\frac{1}{p^2+(\chi_i+\chi_j)/2} =$$

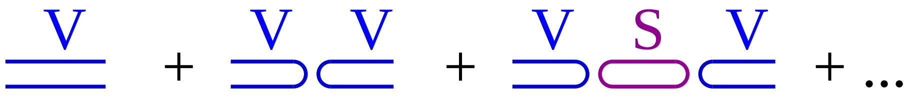


- Neutral propagators have double poles:

$$\mathcal{L}^{(2)} = \sum_{j=1}^{N+2N_V} \epsilon_j(\partial_\mu\Phi_{jj}\partial_\mu\Phi_{jj} + m_j\Phi_{jj}^2) + (m_0^2/N)(\sum_j \epsilon_j\Phi_{jj})^2$$

$$\epsilon_j = \begin{cases} +1 & \text{valence or sea quarks} \\ -1 & \text{ghosts} \end{cases}$$

- Can simply invert with linear algebra tricks. Schematically, for external valence quarks have "hairpin" sum:



Neutral propagator

- Result after $m_0^2 \to \infty$ for $N = 3$ [Bernard & Golterman; Sharpe & Shoresh]

$$\langle \Phi_{ii} \Phi_{jj} \rangle = \frac{\epsilon_i \delta_{ij}}{p^2 + \chi_i} - \frac{1}{N} \frac{1}{(p^2 + \chi_i)(p^2 + \chi_j)} \frac{(p^2 + \chi_{S1})(p^2 + \chi_{S2})(p^2 + \chi_{S3})}{(p^2 + M_{\pi_0}^2)(p^2 + M_\eta^2)}$$

- Simplifies for degenerate sea quarks:

$$\langle \Phi_{ii} \Phi_{jj} \rangle = \frac{\epsilon_i \delta_{ij}}{p^2 + \chi_i} - \frac{1}{N} \frac{(p^2 + \chi_S)}{(p^2 + \chi_i)(p^2 + \chi_j)}$$

  - Manifestly unphysical double pole for $\chi_i = \chi_j$
  - Residue is then $(\chi_i - \chi_S)/N$, so vanishes for physical subspace
  - Can show *from symmetries of PQQCD* that if charged propagators have single poles, then neutral have double (and no higher) poles [Sharpe & Shoresh]

- Propagator becomes physical if i, j are sea quarks, e.g. for degenerate sea

$$\langle \Phi_{SS} \Phi_{SS} \rangle = \frac{1}{p^2 + \chi_S} \left(1 - \frac{1}{N} \right)$$

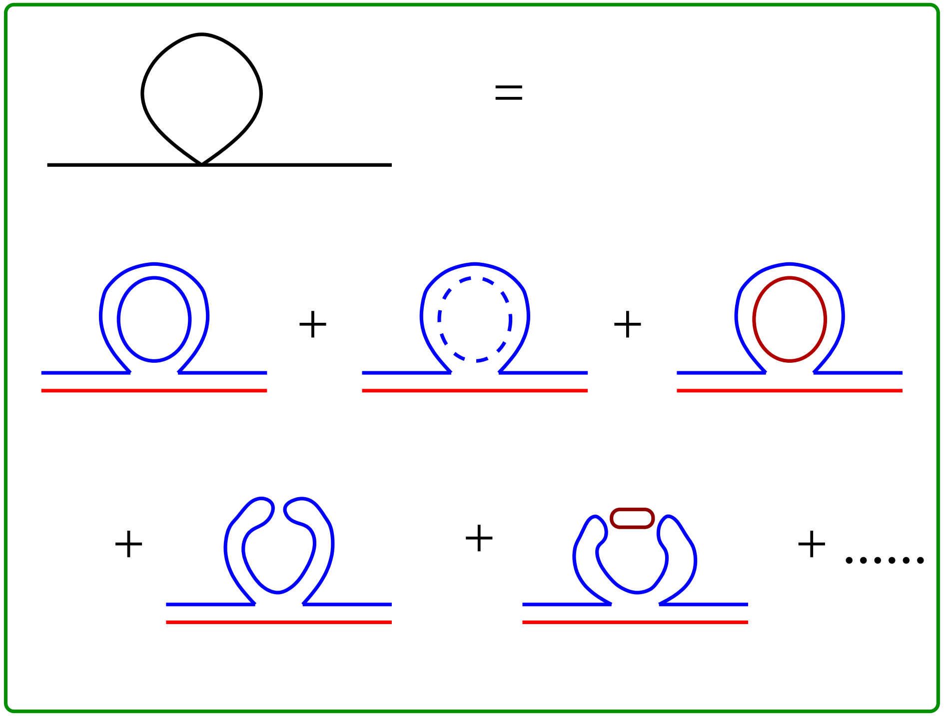
  - Recover projection against η'

Outline of Lecture 3

- Partial quenching and PQχPT
  - What is partial quenching?
  - Developing PQχPT
  - Results and outlook
- $m_u = 0$ and the validity of PQ theories

Sample calculation: m_π^2

- Calculations are straightforward extension of standard χPT
- Mass-squared of "pion" composed of valence quarks $V1, V2$
- Quark-line diagrams for 1-loop contributions



- LO four-pion vertices have single strace, so are "connected"
- Manifest cancellation between contributions from commuting and anticommuting particles

NLO result for m_π^2

- To simplify expression for loop contributions, assume N degenerate sea quarks and $m_{V1} = m_{V2} \neq m_S$

$$m_{VV}^2 = \chi_V \left(1 + \frac{1}{N} \frac{2\chi_V - \chi_S}{\Lambda_\chi^2} \ln(\chi_V/\mu^2) + \frac{\chi_V - \chi_S}{N\Lambda_\chi^2} + \frac{8}{f^2} \left[(2L_8 - L_5)\chi_V + (2L_6 - L_4) N \chi_S \right] \right)$$

  - Reduces to QCD-like result when $\chi_V \to \chi_S$
  - χ_V and χ_S provide separate dials for determining $2L_8 - L_5$ and $2L_6 - L_4$
  - Result in PQ mass-plane depends on physical LECs
  - Unphysical nature of result clear from divergence in $\chi_S \ln \chi_V$ as $\chi_V \to 0$
  - In practice, expansion breaks down only for very small χ_V
- Has been used to determine $2L_8 - L_5$ which, using continuum χPT, constrains physical m_u

Status of PQχPT calculations

- It is now standard to extend any χPT calculation to PQχPT
  - Many quantities considered at NLO: pions, baryons, vector mesons, scalar mesons, heavy-light hadrons, weak matrix elements (B_K, $K \to \pi\pi$), NEDM, pion scattering, ...
  - First calculations at NNLO for pion properties
  - PQ effects also included in tmχPT, staggered χPT and mixed action χPT
  - Most non-trivial example is baryons, where need to use a set-up in which all three quark lines are explicit
  - Most striking result is for scalar meson correlators, where hairpin propagators lead to unphysical *negative* contributions at long distances
- In general, can use PQχPT to determine form of expected results for individual contractions (e.g. connected and disconnected contributions to π_0 propagators in tmLQCD)
- Most extensive practical use is in MILC improved staggered simulations
- Potentially a powerful practical tool, but important to test given incomplete theoretical justification

A final fun example: L_7

$$\mathcal{L}_\chi^{(4)} = \cdots - L_7 \mathrm{str}\,(\chi\Sigma^\dagger - \Sigma\chi^\dagger)^2 + \ldots$$

- Contributes to PGB masses only for non-degenerate quarks
- In QCD, only significant contribution is to m_η

$$4m_K^2 - m_\pi^2 - 3m_\eta^2 = \frac{32(m_K^2 - m_\pi^2)^2}{3f^2}(L_5 - 6L_8 - 12L_7) + \text{chiral logs}$$

- Direct lattice calculation of m_η possible but challenging
- Can we determine L_7 and thus m_η indirectly using PQQCD?
- **Yes, from residue of PQ double pole** [Sharpe & Shoresh]

$$\left.\frac{\int d^3x \langle \Phi_{V1,V1}(t,\vec{x})\Phi_{V2,V2}(0)\rangle}{\int d^3x \langle \Phi_{V1,V2}(t,\vec{x})\Phi_{V2,V1}(0)\rangle}\right|_{m_{V1}=m_{V2}} \underset{t\to\infty}{\longrightarrow} \frac{\mathcal{D}t}{2M_{VV}}$$

- With $N = 3$ degenerate sea quarks find:

$$\mathcal{D} = \frac{\chi_V - \chi_S}{N} - \frac{16}{f^2}\left(L_7 + \frac{L_5}{2N}\right)(\chi_V - \chi_S)^2 + \text{known chiral logs}$$

- PQ simulations allow use of multiple $\chi_V \Rightarrow$ better signal?

Outline of Lecture 3

- Partial quenching and PQχPT
- $m_u = 0$ and the validity of PQ theories

Meaning of "Ambiguity in $m_u = 0$"

- Consider QCD with m_d and m_s fixed (e.g. at their physical values), but send $m_u \to 0$
  - No increase in symmetry
  - $m_\pi^2 \propto (m_u + m_d) + \mathrm{NLO}$ does not vanish
- Contrast this with sending both $m_u, m_d \to 0$:
  - $SU(2)_L \times SU(2)_R$ becomes exact, and $m_\pi^2 \to 0$
- But doesn't $m_u \to 0$ have unambiguous meaning at the level of the lattice action?
  - Naively would seem so if use fermions with exact chiral symmetry (e.g. overlap)
  - But there are (infinitely) many choices for overlap kernel, which assign different topological charges to "rough" configurations
- If we set $m_u = 0$ using two different kernels, will we obtain, in the continuum limit, the same value for mass ratios, e.g. $m_{\pi_0}/m_{\mathrm{proton}}$?
  - The standard answer is **YES**
  - [Creutz, PRL 92, 162003 (2004)] argues **NO!**
- This is the potential ambiguity.

Restate issue in $N_f = 1$ theory

- Can formulate the issue also in $N_f = 1$ QCD, a simpler setting
- No PGBs: spectrum consists of "η", "Δ", etc.
- With two overlap operators having different kernels, if one sets $m = 0$, and takes the continuum limit (not an easy task in practice!) will one get the same value for m_η/m_Δ?
  - The standard answer is **YES**
  - [Creutz, PRL 92, 162003 (2004)] argues **NO**
  - Note that for $a \neq 0$ will certainly have "kernel-dependent" discretization errors—the issue is what happens when $a \to 0$.
- Use this formulation in subsequent discussion:
  - Note that $\langle\bar{\psi}\psi\rangle \neq 0$, although this breaks no symmetry

Standard argument—part I

- In perturbation theory, if have chiral symmetry (as with overlap), quark mass is renormalized multiplicatively, to all orders in PT:

$$
\begin{aligned}
m(a) &= M g(a)^{\gamma_0/\beta_0}[1 + O(g^2)] \\
a\Lambda &= e^{-1/(2\beta_0 g^2)} g^{-\beta_1/\beta_0^2}[1 + O(g^2)] \\
\beta_0 &= (11 - 2N_f/3)/(16\pi^2)
\end{aligned}
$$

- This is uncontroversial. If it were the whole story, it would imply that, once $g(a)$ is small enough (so the universal parts of the β-function and anomalous dimension dominate) setting $M = 0$ ($\Rightarrow m(a) = 0$) leads to universal long-distance physics, irrespective of the overlap kernel.
  - Just as different gauge actions give a Symanzik effective action that differs by $a^2\times$ irrelevant dim-6 operators, so two different $m = 0$ theories will differ by irrelevant dim > 4 operators
- What about non-perturbative contributions to the running?
  - The 't Hooft vertex!

't Hooft vertex contributions

- In one flavor QCD, the 't Hooft vertex is bilinear, and leads to additive shift of quark mass
- Instanton calculations are not reliable when instantons are large, since $g(\rho)$ is not small
- However, what is needed for the RG evolution between scale $1/a$ and $1/(a+da)$ are instantons of size $\rho \sim a$
- If a is small enough, the semi-classical result should be reliable:

$$\begin{aligned}\frac{dm}{d\ln a} &\approx m\gamma_0 g^2 + \mathrm{const} \times (1/a) e^{-8\pi^2/g^2} g^n \\ &\approx m\gamma_0 g^2 + \mathrm{const} \times \Lambda (a\Lambda)^{28/3}\end{aligned}$$

  [Georgi & Macarthy 1981] [Choi, Kim, Sze, PRL 61, 794 (1988)] [Banks, Nir & Seiberg, hep-ph/9403203]

- Additive contribution present, which can only calculate approximately
  - However, it vanishes as $a^{\sim 9}$

Ambiguity or not?

$$\frac{dm}{d\ln a} \approx m\gamma_0 g^2 + \text{const} \times \Lambda(a\Lambda)^{28/3}$$

- There is an uncertainty in the running of m
  - At a given a, for

$$|m(a)| \gtrsim m_{cr} \approx \frac{(a\Lambda)^{28/3}\Lambda}{g(a)^2\gamma_0}$$

    the RG evolution to smaller a will be essentially unaffected by the additive term, and thus unambiguous
  - For $|m(a)| \lesssim m_{cr}$ evolution to smaller a is not controlled
  - In this sense there is an ambiguity in $m(a)$ of size m_{cr}
- As $a \to 0$, however, this ambiguity shrinks rapidly to zero, much faster than the standard logarithmic decrease of $m(a)$
- Thus, in the standard view, we do know, in a regularization invariant way, what $m = 0$ means in the continuum limit
  - In particular, we can simply take $a \to 0$ holding $m(a) = 0$

More on the Ambiguity

- [Creutz, PRL 92, 162003 (2004)] finds this argument unconvincing
- The argument certainly relies on the assumption that we know the form of the non-perturbative terms at short distances
  - Note that the value of $m(a)$ for the massless theory at $a \approx \Lambda_{\rm QCD}^{-1}$ (the "constituent quark mass") is unknown, since the additive term certainly dominates by this scale
  - But this is irrelevant for $m(a)$ as $a \to 0$
- Creutz makes some qualitative arguments, but does not directly address the standard argument given above
  - Please read and draw your own conclusions
- It would be very interesting to test Creutz's proposed breakdown in universality numerically (e.g. in 2-d?)

Relation to PQQCD

- PQ extensions of QCD-like theories provide a way of using symmetries to unambiguously define "$m_u = 0$" [Farchioni *et al.*, 0706.1131,0710.4454]
- Consider the PQ $N_f = 1$ theory, with N_V valence quarks (and corresponding ghosts) *degenerate* with the sea quark
  - Enlarged theory now has an approximate chiral symmetry $SU(N_V + 1|N_V)_L \times SU(N_V + 1|N_V)_R$
  - This symmetry becomes exact when $m \to 0$
  - The fact that $\langle\bar\psi\psi\rangle \neq 0$ in $N_f = 1$ QCD implies that the chiral symmetry of the PQ extension is spontaneously broken
  - One can thus write down the corresponding PQχPT, and $m = 0$ at quark level unambiguously maps to $m = 0$ at the chiral level in order to match the symmetries
  - There are thus PG bosons and fermions with $m_\pi^2 \propto m$
  - Thus $m = 0$ is unambiguously selected by vanishing PQ pion mass, just as $m_u = m_d = 0$ is picked out by vanishing physical pion mass (both requiring $L \to \infty$)
  - Used in practice by [Farchioni, 0710.4454]

More on Relation to PQQCD

- Other (closely related) ways of picking out $m = 0$
  - Vanishing of topological susceptibility, which is defined using PQ correlators [Giusti *et al*, hep-lat/0402027; Lüscher, hep-lat/0404034]
  - $1/m$ divergences in certain *finite volume* PQ correlation functions [Bernard et al, 0711.0696]
- **CONCLUSION:** If $m = 0$ is ambiguous, then the PQ extension of $N_f = 1$ QCD does not have a universal continuum limit
  - For $m = 0$ the PQ pions are massless but m_η, etc. are regularization dependent
- Same argument would apply to other N_f if one of the quark masses vanishes
- These results seem to me to imply that, if $m = 0$ is ambiguous, PQQCD is ill-defined in general (even when $m \neq 0$), and thus that extrapolations using PQχPT are invalid!

Relation to rooting issue

- Rooted staggered fermions, if they are in the correct universality class, give PQQCD in the continuumt limit
  - E.g. for $N_f = 1$, end up with 4 valence and 1 sea quark
- If PQ theories are ill-defined, so is this continuum limit!

Summary

- PQ theories are potentially a very useful practical tool
- They have also been used theoretically, particularly in the ϵ-regime, and to calculate properties of Dirac eigenvalues (e.g. showing that RMT does describe the properties of low eigenvalues)
- Theoretical basis of PQχPT weaker than usual χPT
- The issue of whether "$m_u = 0$" is ambiguous is directly related to the question of whether PQ theories are well defined, and thus deserves further investigation
  - Can the standard arguments that $m_u = 0$ is unambiguous be strengthened, or numerically tested?