Renormalization Group: non perturbative aspects and applications in statistical and solid state physics.

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Saclay, march 3, 2009

Field theory:
- infinitely many degrees of freedom

and
- effective action involving infinitely many vertex functions ($\Gamma^{(n)}$ functions)

BUT...

a finite (and small) number of coupling constants (and masses)!

Question: Why not infinitely many couplings (including the non renormalizable ones)?

Actually, there are infinitely many coupling constants:
Ising model: $V(\phi) \sim \log(\cosh\phi) \sim r\phi^2 + g_0\phi^4 + \dots$

Can we handle them?

Yes, we can!

But it is almost hopeless in perturbation theory $\Longrightarrow$ Wilson RG.
(either Polchinski or Wetterich formalisms)

Wilson RG:

integration on fluctuations scale by scale and not order by order in g_0.
(but needs approximations)

$\Downarrow$

RG is at the center of the formalism (no longer any divergence).

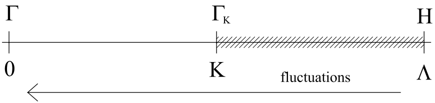
Wilson RG (NPRG)

Idea: Build a one-parameter family of models indexed by a scale k

$$Z[J] \longrightarrow Z_k[J] \quad \text{or for the effective action:} \quad \Gamma[\phi] \longrightarrow \Gamma_k[\phi]$$

that interpolates between the classical model (no fluctuation) at $k = \Lambda$($\sim$inverse lattice spacing) and the quantum one (all fluctuations) at $k = 0$.

- at $k = \Lambda$ all fluctuations are frozen: $\Gamma_{k=\Lambda}[\phi] = S[\phi]$,
- at $k = 0$ all fluctuations are integrated out: $\Gamma_{k=0}[\phi] = \Gamma[\phi]$
- for $0 < k < \Lambda$ the model incorporates only the "rapid" fluctuations: $q \in [k, \Lambda]$.



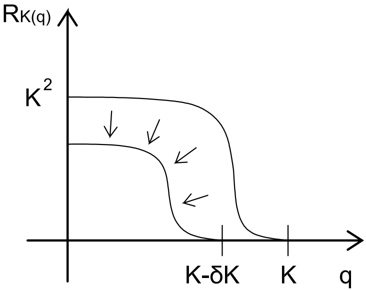
A simple method for freezing the slow modes: give them a large mass $\rightarrow$ modify the original model by adding a momentum dependent mass.

$$Z_k = \int D\varphi \, e^{-S[\varphi]-\Delta S_k[\varphi]} \;\; \text{with} \;\; \Delta S_k[\varphi] = \frac{1}{2}\int_q R_k(q)\,\varphi_q \varphi_{-q}$$

$$\begin{cases} R_{k=\Lambda}(q) = \Lambda^2 & \Longrightarrow \Gamma_{k=\Lambda}[\phi] = S[\phi] \\ R_{k=0}(q) = 0 & \Longrightarrow \Gamma_{k=0}[\phi] = \Gamma[\phi] \end{cases}$$

Lowering k by dk consists in integrating over fluctuations in the shell $[k - dk, k] \Rightarrow$ RG equation:

$$k\partial_k \Gamma_k[\phi] = F[\Gamma_k[\phi]]$$


RK(q)
K2
K-δK
K
q

Wetterich equation

Legendre transform of $W_k[J] = \log Z_k[J]$ (slightly modified)

$$\Gamma_k[\phi] + W_k[J] = \int_q J_q \phi_{-q} - \frac{1}{2} \int_q R_k(q)\, \phi_q \phi_{-q}$$

satisfies

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_q k\partial_k R_k(q) \left[\Gamma_k^{(2)}[q,\phi] + R_k(q)\right]^{-1}$$

Difficult because

- functional,
- partial differential equation,
- non-linear,
- integral;

and

- mass “regulator“ $\Rightarrow$ difficulties with gauge invariance.

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}\int_q k\partial_k R_k(q)\left[\Gamma_k^{(2)}[q,\phi] + R_k(q)\right]^{-1}$$

but. . .

- it looks like a one-loop equation! $\Rightarrow$ tremendous simplification,
- it is a differential version of field theory $\Rightarrow$ solving it with a given initial condition (bare=microscopic action) is a complete solution of the problem,
- it is free of any divergence (the integration is on a small momentum shell thanks to the $\partial_k R_k(q)$ term)
- it is regularized in the infrared by the presence of the scale k $\Rightarrow$ in massless theories the singularities build up as $k \longrightarrow 0$,
- it respects all the symmetries of the problem if the regulator term does.

Local Potential Approximation

Solving Wetterich equation $\Rightarrow$ closure of the infinite hierarchy of equations on the $\Gamma_k^{(n)}$'s.

Idea: If interested only in low momenta (mass gap, phase diagram, thermodynamic quantities) then approximate $\Gamma_k[\phi]$ by a derivative expansion:

$$\Gamma_k[\phi] = \int_x \left(V_k(\phi) + \frac{1}{2} Z_k(\phi)(\nabla\phi)^2 + O(\nabla^4) \right)$$

$\Rightarrow$ Wetterich equation becomes PDF on V_k and Z_k. On top of the derivative expansion: field expansion of $Z_k(\phi)$:

$$Z_k(\phi) = Z_k + O(\phi^2)$$

and even take

$$Z_k(\phi) = 1$$

Local Potential Approximation (LPA) (no field renormalization).

LPA:

$$k\partial_k V_k(\phi) = \frac{1}{2}\int_q \frac{k\partial_k R_k(q)}{Z_k q^2 + R_k(q) + V_k''(\phi)}$$

and for the special choice

$$R_k(q) = Z_k(k^2 - q^2)\,\theta(k^2 - q^2)$$

we obtain

$$k\partial_k V_k(\phi) = \frac{4v_d}{d}\frac{k^{d+2}}{Z_k k^2 + V_k''(\phi)}$$

which is an extremely simple equation although highly non trivial!

The LPA equation

- rules the RG flow of infinitely many couplings:

$$V_k(\phi) = a_{k,0} + a_{k,2}\,\phi^2 + g_{k,4}\,\phi^4 + g_{k,6}\,\phi^6 + \dots$$

- is not based on an expansion in a small coupling,
- is valid in any dimension,
- can be solved with an initial action which is non polynomial $\Rightarrow$ keeps track of all the "microscopic" information $\Rightarrow$ allows us to compute non-universal quantities, that is those that depend on the UV cut-off! (e.g. a critical temperature).
- predicts the convexity of the effective potential in the broken phase (impossible within perturbation theory)**.

Why infinitely many couplings here and only one in perturbation theory ? (and a mass)
Perturbation:

$$\Gamma^{(2n)}(\{p_i\})_{|\mathrm{NP}} \sim g_{2n}$$

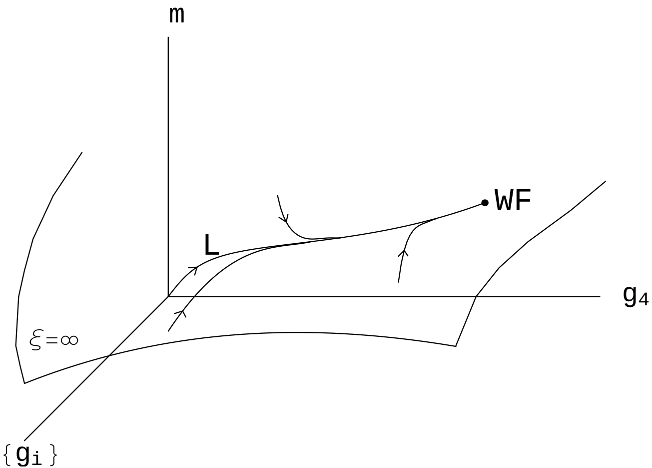
$\Rightarrow$ also infinitely many couplings but all $\Gamma^{(2n)}$ expandable in powers of g_4:

$$g_6 \sim c\, g_4^3 + c' g_4^4 + \dots$$

NPRG:
"large river effect" and more generally

decoupling of high momentum modes

Example: ϕ^4 theory in d=3 (massless case). Asymptotically free theory in the UV:


m
L
WF
g4
ξ=∞
{ gi }

- after a transient regime (lattice effects) all trajectories are very close and are (almost) driven by a single coupling: g_4 (and a mass in the massive case);
- on the trajectory starting from the gaussian, the flow can be reversed in the UV direction and ends at the gaussian fixed point: this theory is UV free;
- almost all memory of the microscopic theory is lost in the IR: universality.

and we can understand

- the importance of asymptotic freedom for the decoupling of rapid modes (or asymptotic safety);
- the meaning of renormalizability and non-renormalizability;
- for $\Lambda \to \infty$ the difference between the perturbative "infinite cut-off limit" and the non perturbative "continuum limit".

Statistical physics

Interest of RG in statistical physics: systems where fluctuations around the "mean field" approximations are large.

- occurence of a second order phase transition (at or out-of thermal equilibrium),
- systems of fermions (or bosons) showing instabilities: superconducting, magnetic, etc...

$O(N)$ models: ferromagnetism

Ferromagnetic systems on a lattice:
(classical) vectors $\vec{S}_i$ of unit norms $|\vec{S}_i| = 1$. Hamiltonian:

$$H = -J \sum_{<i,j>} \vec{S}_i . \vec{S}_j$$

Existence of a phase transition at a critical temperature T_c between

- a symmetric phase: $\langle \vec{S}_i \rangle = 0$
- and a spontaneously broken phase: $\langle \vec{S}_i \rangle = \vec{m}_i \neq 0$

with a spontaneous symmetry breaking pattern:

$$O(N) \to O(N-1)$$

Why field theory?

Because at T_c:

$$\text{correlation length } \xi = m_R^{-1} = \infty$$

$$\Downarrow$$

strongly correlated N-body systems

$$\Downarrow$$

Large fluctuations: violations of the law of large number

$$\Downarrow$$

possibility of non-gaussian theories.

Which field theories?

- nature of the order parameter (ferro: $\vec{m}_i \Rightarrow N$-component vector)
- symmtery breaking pattern ($O(N) \to O(N-1)$)
- power counting.

$$H = \int d^d x \left(\frac{1}{2} (\nabla \vec{\phi})^2 + \frac{1}{2} r \vec{\phi}^{\,2} + g \left(\vec{\phi}^{\,2} \right)^2 \right)$$

$$Z[\vec{J}] = \int D\vec{\phi}\, e^{-H[\vec{\phi}] - \int \vec{J}.\vec{\phi}}$$

Mean Field:

- $r > 0$ symmetric phase: $\langle \vec{\phi} \rangle = 0$
- $r < 0$ broken phase: $\langle \vec{\phi} \rangle \neq 0$

$$\Rightarrow r \propto T - T_c$$

What do we compute with field theory?

1) Interest in universal thermodynamic quantities:

- magnetization: $\vec{M} = \langle \sum_i \vec{S}_i \rangle \propto \langle \int d^d x \, \vec{\phi}(\vec{x}) \rangle$,
- susceptibility: response of $\vec{M}$ to a change of the external source $J =$ magnetic field B: $\chi = \frac{\partial M}{\partial B}$,
- specific heat ,
- behavior of the correlation length ξ,
- equation of state: $f(M, T, B) = 0$, etc...

Generically

$$X \sim (T - T_c)^{-x} \quad \text{with} X = \xi, \chi, M, \ldots$$

$x =$ critical exponents are universal.

$\Rightarrow$ quantities defined from correlation functions at zero momentum.
$\Rightarrow$ computable from the RG flows of the coupling constant, normalization of the field and mass around the fixed point.

2) Interest in non-universal thermodynamic quantities:

- T_c and phase diagram,
- amplitudes $X = A_x(T - T_c)^{-x}$.

Much more difficult to compute perturbatively because dependence on (bare) microscopic details.
Example: T_c of Ising model in $d = 3$, cubic lattice.

3) Interest in the momentum dependence of correlation function(s): $\Gamma^{(2)}(p)$ at and near T_c with or without an external source (magnetic field): also very difficult perturbatively.

Examples:

- Ising, $d = 2$ with a magnetic field: 7 bound states and symmetry E_8!
- Ising, $d = 3$, $T < T_c$ existence of a "bound state"?

How do we compute with field theory?

For second order phase transitions: fixed point(s).

Needs to compute $\beta(g) = \mu \frac{\partial g_R(\mu)}{\partial \mu}$ at high orders. ϕ^4 theory computed

- at five loops in $\epsilon = 4 - d$ expansion,
- at six loops in the zero momentum massive scheme in $d = 3$ (five loops in $d = 2$).

Proof of Borel-summability of renormalized series in $d = 3$ for $(\vec{\phi}^2)^2 \Rightarrow$ an industry about resummation methods $\Rightarrow$ works well for $O(N)$ models in $d = 3$:

N	Resummed pert. exp.		Monte-Carlo	
	η	ν	η	ν
0	0.0284(25)	0.5882(11)	0.030(3)	0.5872(5)
1	0.0335(25)	0.6304(13)	0.0368(2)	0.6302(1)
2	0.0354(25)	0.6703(15)	0.0381(2)	0.6717(1)
3	0.0355(25)	0.7073(35)	0.0375(5)	0.7112(5)
4	0.035(4)	0.741(6)	0.0365(10)	0.749(2)
10	0.024	0.859		

But, $N = 1$, $d = 2$:
$\eta = 0.25$ and at five loops $\eta = 0.145(14)$.

$\eta_{\mathrm{NPRG}} = 0.254$ (good!!).

Out-of equilibrium Statistical Mechanics

Question: How to handle strongly correlated out of equilibrium problems?
For example: Directed Percolation the reaction-diffusion process:

$$\begin{aligned} 2A &\rightarrow A \\ A &\rightarrow 2A \end{aligned} \tag{1}$$

the particles A can diffuse with a diffusion coefficient D.

- Phase transition between active and absorbing state.
- Can we efficiently analyze the long time and large distance behavior?
- A widespread typical problem, very hard to handle both analytically and numerically.

Non-linear Langevin equations

- Non-linear Langevin equation

$$\partial_t \varphi(\vec{x}, t) = F[\varphi] + G[\varphi]\zeta(\vec{x}, t),$$

where ζ is a local, Gaussian white noise:

$$\begin{aligned} \langle \zeta(\vec{x}, t) \rangle &= 0, \\ \langle \zeta(\vec{x}, t)\zeta(\vec{x}', t') \rangle &= 2\delta^{(d)}(\vec{x} - \vec{x}')\delta(t - t'). \end{aligned} \tag{2}$$

Leads to field theories in a way analogous to stochastic quantization.

But... when no fluctuation-dissipation theorem $\Rightarrow$ the probability distribution of stationary states is unknown.

Examples:

1. *Directed Percolation universality class*
   A realization: the reaction-diffusion process.

- Can be described by the Langevin equation with multiplicative noise:

$$\partial_t \varphi(\vec{x}, t) = D\nabla^2\varphi + \sigma\varphi - \lambda\varphi^2 + \mu\sqrt{\varphi}\zeta(\vec{x}, t) \qquad (3)$$

2. *Kardar-Parisi-Zhang equation*
   Takes the form

$$\partial_t \varphi(\vec{x}, t) = \nu\nabla^2\varphi + \frac{\lambda}{2}(\nabla\varphi)^2 + \sigma\zeta(\vec{x}, t) \qquad (4)$$

- Describes kinetic roughening of a d-dimensional interface among many other phenomena.
- Shows generic scaling: no fine-tuning.
- Mean-field-like approximations fail to describe the rough phase.

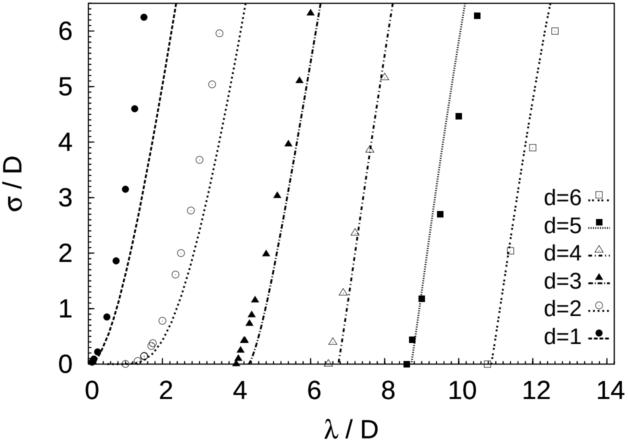
3 *Model A*

- describes (one of) the dynamics of the Ising model
- satisfies detailed balance (fluctuation-dissipation theorem)
- known probability distribution of stationary states: Gibbs
- $\partial_t \phi(t, \vec{x}) = -\frac{\delta H}{\delta \phi(t, \vec{x})} + \zeta(t, \vec{x})$

Powerful supersymmetric methods. Very good perturbative and non-pertubative results.

- Critical exponents:

d		(a) LPA [1]	(b) LPA' [1]	(c) MC [2]
3	ν	0.584	0.548	0.581(5)
	β	0.872	0.782	0.81(1)
	z	2	1.909	1.90(1)
2	ν	0.730	0.623	0.734(4)
	β	0.730	0.597	0.584(4)
	z	2	1.884	1.76(3)
1	ν	1.056	0.888	1.096854(4)
	β	0.528	0.505	0.276486(8)
	z	2	1.899	1.580745(10)


σ / D
λ / D
0
1
2
3
4
5
6
0
2
4
6
8
10
12
14
d=6
d=5
d=4
d=3
d=2
d=1

Disordered systems

Question: relevance of disorder in statiscal systems.

Most famous problems:

- Random Field Ising Model: dimensional reduction true at all orders of perturbation theory, but... wrong.
- Spin glasses.

Quantum fermionic systems

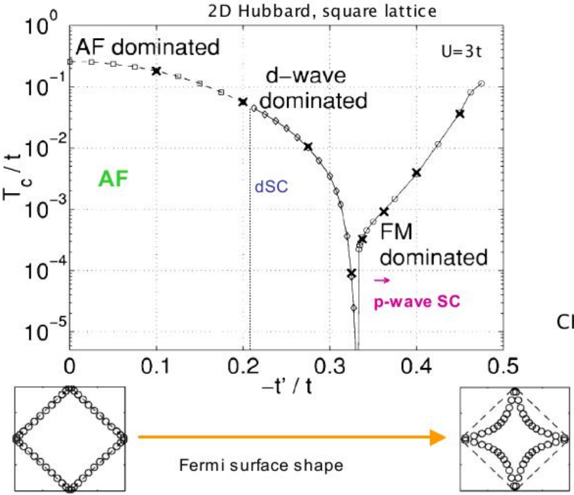
Specific difficulties:

- microscopic degrees of freedom $=$ fermions (electrons) $\Rightarrow$ Fermi surface
- but... order parameters $=$ bilinear in the fields (e.g. superconductivity)
- competition between different kinds of instabilities: superconductivity, charge (or spin) density waves corresponding to different kinds of bilinear: $\langle\psi^\dagger\psi^\dagger\rangle$ or $\langle\psi^\dagger\psi\rangle$,
- suscpetibility $=$ four point function, depends on three momenta around the Fermi surface $\Rightarrow$ renormalization of a full momentum dependent function $\Rightarrow$ functional renormalization.

Crucial advantage: it is unbiased! The RG flow chooses in which phase the system ends up.

Most famous example: the Hubbard model

$$H = -t \sum_{nn,s} c^\dagger_{i,s} c_{j,s} - t' \sum_{nnn,s} c^\dagger_{i,s} c_{j,s} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$



CH& Salmhofer, PRL 2001