Lattice QCD and Non-perturbative Renormalization

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Lecture 1: generalities, lattice regularizations, Ward-Takahashi identities

Quantum Field Theory and Divergences in Perturbation Theory

- A local QFT has no *small* fundamental lenght: the action depends only on products of fields and their derivatives at the same points. In perturbation theory (PT), propagator has simple power law behavior at short distances and interaction vertices are constant or differential operators acting on δ-functions.

- Perturbative calculation affected by divergences due to severe short distance singularities. Impossible to define in a direct way QFT of point like objects.

- The field ϕ_i has a momentum space propagator (in d dimensions)

$$\Delta_i(p) \sim \frac{1}{p^{\sigma_i}} \quad \text{as} \quad p \to \infty \qquad \Rightarrow \qquad [\phi_i] \equiv \frac{1}{2}(d - \sigma_i) \quad (\textit{canonical dimension})$$

- $[\phi] = \frac{1}{2}(d - 2 + 2s)$ for fields of spin s.
  It coincides with the natural mass dimension of ϕ for $s = 0, \frac{1}{2}$.

- Dimension of the type α vertices $V_\alpha(\phi_i)$ with n_i^α powers of the fields ϕ_i and k_α derivatives:

$$\delta[V_\alpha(\phi_i)] \equiv -d + k_\alpha + \sum_i n_i^\alpha [\phi_i]$$

- A Feynman diagram γ represents an integral in momentum space which may diverge at large momenta. Superficial degree of divergence of γ with L loops, I_i internal lines of the field ϕ_i and v_α vertices of type α:

$$\delta[\gamma] = dL - \sum_i I_i \sigma_i + \sum_\alpha v_\alpha k_\alpha$$

- Two topological relations

$$E_i + 2I_i = \sum_\alpha n_i^\alpha v_\alpha \qquad \text{and} \qquad L = \sum_i I_i - \sum_\alpha v_\alpha + 1$$

$$\Rightarrow \qquad \boxed{\delta[\gamma] = d - \sum_i E_i[\phi_i] + \sum_\alpha v_\alpha \delta[V_\alpha]}$$

- Classification of field theories on the basis of divergences:

1. Non-renormalizable theories. $\exists\ i \mid \delta[V_i] > 0 \Rightarrow$ diagrams with increasing number v_i of vertices V_i may have arbitrarily large degree of divergence.
2. Super-renormalizable theories. A finite number of diagrams is superficially divergent ($\delta[V_i] < 0,\ \forall\ i$).
3. Renormalizable theories. $\delta[V_i] \leq 0,\ \forall\ i \Rightarrow$ only a finite number of sets of external lines can yield supeficially divergent diagrams.

Regularization methods

- Due to divergences, QFT can not be defined directly in PT.

- Strategy: modify the theory at large momentum, short distance (e.g. use a cut-off at large momenta) or otherwise in such a way that Feynman diagrams become well-defined finite quantities and when some parameter reaches some limit (e.g. cut-off sent to infinity), one formally recovers the original PT.

  1. Cut-off (Pauli Villar's) regularization: modify the propagator in such a way that it decreses faster at large momentum. e.g. for a scalar field theory

     $$(p^2+m^2)^{-1} \to \left(p^2+m^2+\alpha_2\frac{p^4}{\Lambda^2}+\alpha_3\frac{p^6}{\Lambda^4}+\ldots+\alpha_n\frac{p^{2n}}{\Lambda^{2n-2}}\right)^{-1}$$

     and choose n to make all diagrams convergent. Λ is the cut-off and when $\Lambda\to\infty$ the original propagator is recovered. For fermions:

     $$(m+i\ \not{p})^{-1} \to \left[m+i\ \not{p}\left(\alpha_1\frac{p^2}{\Lambda^2}+\ldots+\alpha_n\frac{p^{2n}}{\Lambda^{2n}}\right)\right]^{-1}$$

     It does not work for theories in which the action has a definite geometric character like gauge theories.

2. Dimensional regularization: analytic continuation of Feynman diagrams to arbitrary complex values of the dimension d. In 4 dimensions, divergences appear as poles in $4-d$. In theories with fermions, possible subtleties are related with the treatment of γ_5 for generic d (there are at least three different ways to deal with this problem: NDR, DR, HV). It preserves all the symmetries of gauge theories and leads to the simplest perturbative calculations. However it is defined only in perturbation theory.

3. Lattice regularization: time and space are discretized by putting the theory on an hypercubic lattice $\mathbb{Z}^d$ with lattice spacing a. The dynamical variables are the values of the field on the lattice sites (for fermion and boson fileds) or on the link joining two sites (for the gauge fields). Field derivatives are replaced by finite differences. The theory is modified at short distances and a^{-1} acts as an UV cut-off. The lattice regularization preserves most of the global and local symmetries with the obvious exception of the space-time $O(d)$ symmetry which is replaced by an hypercubic symmetry. In theories with fermions there are also problems with chiral symmetry. It is the only known non-perturbative regularization. The regularized functional integral can be calculated by non-perturbative methods as e.g. stochastic methods (Monte Carlo) or strong coupling expansions.

Perturbative Renormalization

- Simple example: scalar field ϕ with $V(\phi) = g\phi^4$ interaction ($\delta[V] = 0$)

$$S(\phi) = \int d^d x \left(\tfrac{1}{2}\partial_\mu \phi \partial_\mu \phi + \tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{4!} g \phi^4\right)$$

- To give a meaning to perturbation theory we replace the action $S(\phi)$ by a regularized action $S_\Lambda(\phi)$ (called bare action) for instance by introducing a momentum cut-off $\Lambda \Rightarrow (m^2 - \Delta) \rightarrow (m^2 - \Delta)_\Lambda = m^2 - \Delta + \alpha_2 \frac{\Delta^2}{\Lambda^2} - \alpha_3 \frac{\Delta^3}{\Lambda^4} + \ldots$

- We introduce two *renormalized* parameters m_r and g_r. In $d = 4$, since $\delta[V] = 0$, one can prove that it is possible to rescale

$$\phi \rightarrow Z^{1/2}(\Lambda, m_r, g_r)\phi_r \qquad\qquad \phi_r \equiv \text{ renormalized field}$$

by choosing the bare parameter m and g as function of m_r, g_r and Λ such that all ϕ_r correlation functions have a finite limit, order by order, in PT when $\Lambda \rightarrow \infty$ with m_r, g_r fixed. For this reason the theory is called renormalizable.

- We introduce the notion of renormalized action $S_r(\phi_r)$:

$$S_\Lambda(\phi) \equiv S_r(\phi_r) = \int d^4x \, \big[\tfrac{1}{2}\phi_r(m_r^2 - \Delta)_\Lambda \phi_r + \tfrac{1}{4!} g_r \phi_r^4$$
$$+ \tfrac{1}{2}(Z-1)\partial_\mu \phi_r \partial_\mu \phi_r + \tfrac{1}{2}\delta m^2 \phi_r^2 + \tfrac{1}{4!} g_r (Z_g - 1)\phi_r^4\big]$$

where $(m_r^2 - \Delta)_\Lambda$ refers to the regularization.

- Identity between $S_r(\phi_r)$ and $S_\Lambda(\phi)$ expressed by the set of relations:

$$\phi = Z^{1/2}\phi_r, \qquad g = g_r Z_g / Z^2, \qquad m^2 = (m_r^2 + \delta m^2)/Z$$

- $S_r(\phi_r) = \text{tree} - \text{level} + \text{counterterms}$, where the counterterms are parametrized in terms of the renorm. constants Z, Z_g, δm^2 which are formal series in g_r:

$$\delta m^2 = a_1(\Lambda) g_r + a_2(\Lambda) g_r^2 + \ldots$$
$$Z = 1 + b_1(\Lambda) g_r + b_2(\Lambda) g_r^2 + \ldots$$
$$Z_g = 1 + c_1(\Lambda) g_r + c_2(\Lambda) g_r^2 + \ldots$$

- Owing to $\delta[V]=0$ in $d=4$, the superficial degree of divergence of Feynman diagrams is independent of the order in PT and one can prove that $a_n(\Lambda)$, $b_n(\Lambda)$ and $c_n(\Lambda)$ can be chosen such that all correlation functions have a finite large Λ limit, order by order in g_r.

- DIGRESSION: renormalization formalism is better expressed in terms of the 1 partcle-irreducible (1PI) generating functional $\Gamma(\varphi)$ (also called generating funcional of proper vertices). To obtain it, we have to start from the generating functional of correlation functions:

$$Z(J)=\int[d\phi]\exp(-S(\phi)+J\phi) \qquad \Rightarrow$$

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle=\frac{1}{Z(J)}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z(J)\Big|_{J=0}$$

from which we define the generating functional of connected correlation functions $W(J)=\ln Z(J)$. $W(J)$ is convex $\Rightarrow$ has a Legendre transform

$$\Gamma(\varphi)=\sup_J\left[\int dx J(x)\varphi(x)-W(J)\right] \quad \Rightarrow \quad \varphi(x)=\frac{\delta W}{\delta J(x)} \quad \Rightarrow \quad J_{\text{stat}}(x,\varphi)$$

$$\Gamma(\varphi)=\sum_n\frac{1}{n!}\int dx_1\cdots dx_n\Gamma^{(n)}(x_1,\ldots,x_n)\varphi(x_1)\cdots\varphi(x_n)$$

- One can show that $\Gamma^{(n)}(x_1, \ldots, x_n)$ are the 1PI correlators. In particular, $\Gamma^{(2)}(x_1, x_2) = [W^{(2)}(x_1, x_2)]^{-1}$ is the inverse propagator of the field ϕ.

- We can define $\Gamma_r(\varphi_r)$ as in the case of $S_r(\phi_r)$ and expand it series of g_r

$$\Gamma_r(\varphi_r) = \Gamma_0(\varphi_r) + g_r \Gamma_1(\varphi_r) + O(g_r^2)$$

  where it turns out that $\Gamma_0(\varphi_r) = \lim_{\Lambda \to \infty} S_\Lambda(\phi_r)$ is the tree-level action and $\Gamma_1(\varphi_r)$ contains the counterterms at one-loop (and thus Z, Z_g and δm^2).

- The insertion of composite operators in correlation functions of elementary fields require additional RCs (one for each new operator). This can be proven by adding a new source t (coupled to the composite operator) to $Z(J) \rightarrow Z(J, t)$ and computing again the renormalized 1PI generating functional $\Gamma_r(\varphi, t)$. Functional derivatives in t gives the insertions of the new operator and one obtain the relations between bare and renormalized proper vertices.

- General result of renormalization theory: a composite operator $\mathcal{O}(\phi)$ with canonical dimension $[\mathcal{O}(\phi)] = D$ mixes under renormalization with all the (composite) operators of equal or lower dimension allowed by symmetries:

$$[\mathcal{O}(\phi(x))]_r = \sum_{\alpha:[\mathcal{O}_\alpha]\leq D} Z_\alpha \mathcal{O}_\alpha(\phi(x))$$

The coefficients Z_α of the mixing with $\mathcal{O}_\alpha$ (with $[\mathcal{O}_\alpha] = D$) are at most logarithmically divergent. If the regularization is performed through an hard cut-off Λ (e.g. Pauli-Villar's or lattice) $\Rightarrow$ mixing coefficients Z_β of the lower dimensional operators $\mathcal{O}_\beta$ may contain power divergences $\sim \Lambda^{(D-[\mathcal{O}_\beta])}$.

- Imposing correlation functions to be finite at some given order in PT only determines the divergent part of the renormalisation constants (RCs) Z, Z_g and δm^2 at that order (e.g. $a_1(\Lambda)$, $b_1(\Lambda)$ and $c_1(\Lambda)$ at one-loop).

- One can add them arbitrary finite constants. These are fixed through the choice of the renormalization conditions which results in the renorm. scheme dependence of renorm. constants and of renormalized field and parameters. This dependence can be shown to cancel from any physical observable.

- For instance, in the present case we can choose the three following conditions in momentum space (consistent with the tree-level approximation):

$$\tilde{\Gamma}_r^{(2)}(p=0) = m_r^2, \qquad \frac{\partial}{\partial p^2}\tilde{\Gamma}_r^{(2)}(p)\Big|_{p=0} = 1, \qquad \tilde{\Gamma}^{(4)}(0,0,0,0) = g_r$$

- These conditions detrmines completely the renormalization constants:

$$a_1(\Lambda) = \frac{-1}{16\pi^2}\left(\frac{\Lambda^2}{2} - m_r^2 \ln\frac{\Lambda}{m_r}\right) + O(1), \quad b_1(\Lambda) = \frac{3}{16\pi^2}\ln\frac{\Lambda}{m_r} + O(1), \quad c_1(\Lambda) = 0$$

- Proving the renormalization of a theory at any order of PT is not an easy task. The 2-point function is quadratically divergent and the 4-point function logarithmically divergent at any order in PT as power counting shows. However, beyond one-loop a new difficulty arises: superficially convergent diagrams have divergent sub-diagrams. Some of them can be made finite through the insertion at higher order of counterterms of lower order. Some other diagrams present the problem of the so-called overlapping divergences and in order to prove that also them can be made finite a more sophisticated technique has been developed (the so called BPHZ prescription).

- To conclude, one can prove that in field theories with vertices V_α such that $\delta[V_\alpha] \leq 0, \forall\alpha$ all the correlation functions can be made finite by choosing a finite number of counterterms as function of the renormalized parameters and of the regularization method used.

- In field theories for which $\exists\, \alpha \mid \delta[V_\alpha] > 0$, the number of counterterms needed increases with the order of PT and thus is infinite. These theories are therefore called non-renormalizables. This is the case of Effective Field Theories (EFT) which represent the approximation in some limit of an underlying, more fundamental, theory. Two examples are:

  1. Chiral Perturbation Theory (ChPT), i.e. the effective theory of QCD at small momenta and quark masses. It describes interactions of light hadrons by automatically encoding the constraints coming from the pattern of spontaneous chiral symmetry breaking. The parameters of the expansion are momenta and masses of the corresponding Goldstone bosons.
  2. Heavy Quark Effective Theory (HQET), i.e. the effective theory which describes the interactions of an heavy quark with other light quarks. It is based on an expansion of QCD in inverse powers of the heavy quark mass.

- In the case of ChPT, the interaction of light mesons is completely perturbative in their masses and momenta (the strong interacting quark fields are not anymore the elementary degrees of freedom of the theory as instead in QCD).

- On the contrary, HQET is perturbative in the (inverse) heavy quark mass but is still non-perturbative in the strong interaction between the light quarks and the heavy quark, at a given order in heavy quark mass expansion. We will present the case of non-perturbative HQET later in this course.

- Non-renormalizable theories are still very useful in the limit for which they have been conceived. If one can fix all the counterterms at a given order in the expansion by using a certain number of observables, then the theory can predict the value of other observables at that order.

Renormalization Group Equations

- Relation between bare and renormalized n-point 1PI correlation functions

$$\tilde{\Gamma}_r^{(n)}(p_1,\ldots,p_n;g_r,m_r,\Lambda) = Z(g_r,\tfrac{m_r}{\Lambda})^{(n/2)}\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n;g,m,\Lambda)$$

Take the logarithmic derivative of $\Gamma_r^{(n)}$ with respect to m_r at g, Λ fixed

l.h.s.: $$m_r\frac{\partial}{\partial m_r}\tilde{\Gamma}_r^{(n)}\Big|_{g,\Lambda} = m_r\frac{\partial}{\partial m_r}\tilde{\Gamma}_r^{(n)} + m_r\frac{\partial g_r}{\partial m_r}\Big|_{g,\Lambda}\frac{\partial}{\partial g_r}\tilde{\Gamma}_r^{(n)}$$

r.h.s.: $$m_r\frac{\partial}{\partial m_r}(Z^{(n/2)}\tilde{\Gamma}^{(n)})\Big|_{g,\Lambda} = \frac{n}{2}m_r\frac{\partial\ln Z}{\partial m_r}\Big|_{g,\Lambda}\tilde{\Gamma}_r^{(n)} + Z^{(n/2)}\,m_r\frac{\partial}{\partial m_r}\tilde{\Gamma}^{(n)}\Big|_{g,\Lambda}$$

Putting them together on obtain the Callan-Symanzik equation

$$\left[m_r\frac{\partial}{\partial m_r} + \beta_r(g_r,\tfrac{m_r}{\Lambda})\frac{\partial}{\partial g_r} - \frac{n}{2}\eta_r(g_r,\tfrac{m_r}{\Lambda})\right]\tilde{\Gamma}_r^{(n)} = (2-\eta_r(g_r,\tfrac{m_r}{\Lambda}))m_r^2\tilde{\Gamma}_r^{(1,n)}$$

where $\qquad \beta_r(g_r, \frac{m_r}{\Lambda}) \equiv m_r \frac{\partial g_r}{\partial m_r}\Big|_{g,\Lambda}, \qquad \eta_r(g_r, \frac{m_r}{\Lambda}) \equiv m_r \frac{\partial \ln Z}{\partial m_r}\Big|_{g,\Lambda}$

and the r.h.s. term $\tilde{\Gamma}_r^{(1,n)}$ corresponds to the insertion of ϕ_r^2 at zero momentum in $\tilde{\Gamma}_r^{(n)}$. Moreover, if the $\Gamma_r^{(n)}$ have a finite limit when $\Lambda \to \infty$ than β_r and η_r have also such a limit where they become functions of g_r only.

- The Callan Symanzik equation can be used to prove renormalizability inductively on the number of loops in PT.

- Bare RG equation. Detailed prturbative analysis shows that

$$\tilde{\Gamma}_r^{(n)}(p_i; g_r, m_r, \Lambda) = \tilde{\Gamma}_r^{(n)}(p_i; g_r, m_r) + O(\Lambda^{-2}(\ln \Lambda)^L)$$

Differentiating $\tilde{\Gamma}_r^{(n)}(p_i; g_r, m_r, \Lambda)$ with respect to Λ at g_r, m_r fixed $\Rightarrow$

$$\Lambda \frac{\partial}{\partial \Lambda}(Z^{(n/2)}(g, \frac{m_r}{\Lambda})\tilde{\Gamma}^{(n)}(p_i; g, m, \Lambda))\Big|_{g_r, m_r} = O(\Lambda^{-2}(\ln \Lambda)^L)$$

- We neglect here terms subleading by powers of Λ (called *scaling violations*)

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\eta(g)-\eta_2(g)(m^2-m_c^2)\frac{\partial}{\partial m^2}\right]\tilde{\Gamma}^{(n)}(p_i;g,m,\Lambda)=0$$

$$\beta(g)\equiv\Lambda\frac{\partial g}{\partial\Lambda}\Big|_{g_r,m_r}\quad \eta(g)\equiv-\Lambda\frac{\partial\ln Z}{\partial\Lambda}\Big|_{g_r,m_r}\quad \eta_2(g)\equiv-\Lambda\frac{\partial\ln Z_2/Z}{\partial\Lambda}\Big|_{g_r,m_r}$$

where Z_2 is a new renormalization constant for the composite operator ϕ^2 and where β, η and η_2 can be shown to be independent of m_r/Λ in the large Λ limit (up to some correction which goes to zero like some power of m_r/Λ).

- We want to relate $\beta_r(g_r)=m_r\frac{\partial g_r}{\partial m_r}\Big|_{g,\Lambda}$ and $\beta(g)=\Lambda\frac{\partial g}{\partial\Lambda}\Big|_{g_r,m_r}$. For dimensional reasons $g_r=g_r(g,m_r/\Lambda)$ and thus

$$0=\Lambda\frac{\partial g_r}{\partial\Lambda}\Big|_{g_r,m_r}=\left(\beta(g)\frac{\partial}{\partial g}+\Lambda\frac{\partial}{\partial\Lambda}\right)g_r(g,m_r/\Lambda)=\left(\beta(g)\frac{\partial}{\partial g}-m_r\frac{\partial}{\partial m_r}\right)g_r(g,m_r/\Lambda)$$

$\Rightarrow\ \beta(g)\frac{\partial g_r}{\partial g}=\beta_r(g_r)$ i.e. the sign and the zeros of β and β_r are the same.

- β and β_r can be used to understand the behaviour of g and g_r when the cut-off Λ is removed (i.e. in the scaling region $\Lambda\to\infty$) which, in the case of the lattice regularization, corresponds to the continuum limit ($a\sim 1/\Lambda\to 0$).

Fixed Points

- Equation for g_r (with $a \sim \frac{1}{\Lambda}$):

$$\beta_r(g_r) = m_r \frac{\partial g_r}{\partial m_r}\Big|_{g,\Lambda} = m_r a \frac{\partial g_r}{\partial m_r a}\Big|_{g,\Lambda} \quad \text{with solution} \quad \ln \frac{m_r a}{\overline{m}_r a} = \int_{\bar{g}_r}^{g_r} \frac{dg}{\beta_r(g)}$$

- Consider the case in which β_r has two simple zeros: one with positive slope in g_1 and the other with negative slope in g_2 ($\beta_r(g_r) > 0$ for $g_1 < g_r < g_2$).

1. If $g_1 > \bar{g}_r$, $m_r a \longrightarrow 0$ at fixed $g \;\Rightarrow\; g_r$ increases $\longrightarrow g_1$.
2. If $g_1 < \bar{g}_r < g_2$, $m_r a \longrightarrow 0$ at fixed $g \;\Rightarrow\; g_r$ decreases $\longrightarrow g_1$.
3. If $\bar{g}_r > g_2$, $m_r a \longrightarrow 0$ at fixed $g \;\Rightarrow\; g_r$ increases away from g_2.

- Similarly, equation for g:

$$\beta(g) = -a \frac{\partial g}{\partial a}\Big|_{g_r, m_r} = -m_r a \frac{\partial g}{\partial m_r a}\Big|_{g_r, m_r} \quad \text{with solution} \quad \ln \frac{m_r a}{\overline{m}_r a} = -\int_{\bar{g}}^{g} \frac{dg'}{\beta(g')}$$

- Owing to the fact that β has tha same sign and the same zeros of β_r:

1. If $g_1 > \bar{g}$, $m_r a \rightarrow 0$ at fixed $g_r \;\Rightarrow\; g$ decreases away from g_1.
2. If $g_1 < \bar{g} < g_2$, $m_r a \rightarrow 0$ at fixed $g_r \;\Rightarrow\; g$ increases $\rightarrow g_2$.
3. If $\bar{g} > g_2$, $m_r a \rightarrow 0$ at fixed $g_r \;\Rightarrow\; g$ decreases $\rightarrow g_2$.

- At fixed g, g_r driven toward g_1 and away from g_2.
  At fixed g_r, g driven toward g_2 and away from g_1.
  g_1 called Infra-Red (IR) fixed point while g_2 Ultra-Violet (UV) fixed point.

- UV fixed points yield the possibility of continuum limits with a variety of g_r. IR fixed points determine these bounds on g_r. Defining a continuum limit away from an UV fixed point, g_r will approach an IR fixed point.

- In the $g\phi^4$ theory, $g = 0$ is an IR fixed point ($Gaussian$ fixed point) and every continuum limit in this domain will have $g_r = 0$ i.e. a non-interacting theory. The possibility of a non-trivial continuum limit require an UV fixed point which however has been proven not to exist in the whole range $g \in [0, \infty]$

- In pure Yang-Mills, $g = 0$ is an UV fixed point ($asymptotic\ freedom$) and thus for fixed g_r the continuum limit is obtained by sending the bare coupling to zero. The solution of the equation for g by using $\beta(g)$ at two-loop is

$$a = \frac{1}{\Lambda_{LAT}} \exp\left(-\frac{1}{2\beta_0 g^2}\right)(\beta_0 g^2)^{-\beta_1/(2\beta_0^2)}\{1 + O(g^2)\}$$

  whose integration constant defines the mass scale Λ_{LAT}. Λ_{LAT} appears despite the fact that gauge theories do not contain any mass scale.

- $\Rightarrow$ Every physical quantity P with dimensions of a mass is proportional to Λ_{LAT}, i.e. $P = C_P \Lambda_{LAT}$ and P satisfies the RG equation

$$\left[a\frac{\partial}{\partial a} - \beta(g)\frac{\partial}{\partial g}\right]P = O(a^2)$$

- For two such quantities P_1 and P_2, the ratio $\frac{P_1}{P_2} = \frac{C_{P_1}}{C_{P_2}}\{1 + O(a^2)\}$ is constant up to small $O(a^2)$ artifacts. This is called the scaling region.
- DIGRESSION: continuum massive QCD. To understand the (perturbative) renormalization properties of the continuum theory, dimensional regularization is the most suited. The action read:

$$S(A_\mu, \bar{q}_i, q_i) = -\int d^4x \left[\frac{1}{4}\mathrm{Tr}F_{\mu\nu}^2 + \sum_i \bar{q}_i[\gamma_\mu(\partial_\mu - igA_\mu) + m_i]q_i\right]$$

$+$ ghost and gauge fixing term. We renormalize the action through:

$$A_\mu = Z_3^{1/2} A_\mu^r, \quad q_i = Z_q^{1/2} q_i^r, \quad m_i = Z_m m_i^r, \quad g = Z_g g_r \mu^\epsilon$$

where $\epsilon = 4 - d$ and μ is an arbitrary mass scale introduced to keep g_r dimensionless in arbirary dimension d.

- The bare parameters g and m are μ-independent $\Rightarrow g_r = g_r(\mu)$. Through the $Z_i = Z_i(g_r)$ factors also m_i^r become μ-dependent (but they are not observable). The μ-dependence should cancel in observable quantities.

- The Callan-Symanzik equation for a (μ-independent) physical observable read

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_r(g_r)\frac{\partial}{\partial g_r} - \gamma_r(g_r)m_r\frac{\partial}{\partial m_r}\right] P(x; g_r, m_r, \mu) = 0$$

$$\beta_r(g_r) \equiv \mu\frac{\partial g_r}{\partial\mu}\Big|_{m_r} \quad \gamma_r(g_r) \equiv -\frac{\mu}{m_r}\frac{\partial m_r}{\partial\mu}\Big|_{g_r}$$

where one can choose a mass-independent renormalization scheme in which β_r and γ_r are mass independent.

- The Callan-Symanzik equation has two independent standard solutions called RG invariants (RGI): the Λ_{QCD}-parameter and the RGI quark mass M

$$\Lambda_{QCD} = \mu\ (\beta_0 g_r^2)^{-\beta_1/(2\beta_0^2)} e^{\left(-\frac{1}{2\beta_0 g_r^2}\right)} \exp\left(\int_0^{g_r} dh\left[\frac{1}{\beta_r(h)} + \frac{1}{\beta_0 h^3} - \frac{\beta_1}{\beta_0^2 h}\right]\right)$$

$$M = m_r(2\beta_0 g_r^2)^{-\gamma_0/(2\beta_0)} \exp\left(\int_0^{g_r} dh\left[\frac{\gamma_r(h)}{\beta_r(h)} + \frac{\gamma_0}{\beta_0 h}\right]\right)$$

- Every physical quantity in massive QCD is function of Λ_{QCD} and M.

Lattice Regularization of QCD

- Let's consider the naive discretization of the free fermion action:

$$S_{\text{naive}} = a^4 \sum_x \left\{ \frac{1}{2a} \sum_\mu [\bar\psi(x)\gamma_\mu\psi(x+\mu) - \bar\psi(x)\gamma_\mu\psi(x-\mu)] + m\bar\psi(x)\psi(x) \right\}$$

$$= (2\pi)^4 \int d^4p\, \bar\psi(-p) \Big(i \sum_\mu \gamma_\mu \tfrac{\sin a p_\mu}{a} + m \Big) \psi(p)$$

- Problem with the corresponding propagator: $\frac{\sin a p_\mu}{a} \sim p_\mu + O(a^2)$ for $a \to 0$ in the neighborood of $p_\mu = 0$ and π/a. So in this formulation there are 2^4 fermions (*fermion doubling* problem).

- Wilson regularization: add the term

$$S_W \;=\; S_{\text{naive}} - \frac{a^3 r}{2} \sum_{\mu,x} [\bar\psi(x)\psi(x+\mu) + \bar\psi(x+\mu)\psi(x) + 2\bar\psi(x)\psi(x)]$$

$$\sim \int d^4x \bar\psi[\gamma_\mu\partial_\mu + m]\psi - \tfrac{ar}{2} \int d^4x \bar\psi \partial^2 \psi + O(a^2) \quad \text{as} \quad a \to 0$$

- Now, for each component p_μ close to π/a, the mass is increased by $r/a \Rightarrow$ $2^d - 1$ spurious states disappear in the continuum limit (usually r is set to 1).

- By adding the coupling to the gauge links and the pure gauge action one obtains Wilson regularizatilon of QCD.

- The Wilson term is an *irrelevant* operator do dimension 5 which disappears at tree-level in the continuum limit. Its presence breaks explicitly chiral symmetry which is recovered at tree-level for $a \to 0$ (if the soft-breaking term $m = 0$).

- However, at higher orders in PT, the factor a in front of the Wilson term is compensated by $1/a^p$ divergences in the loops, leading to finite or divergent contributions. The formal chiral properties of QCD are lost: $m \to 0$ does not correspond to the chiral limit but there is an additive mass renormalization and operators belonging to different chiral representations mix among themselves.

- Fermion doubling and chiral symmetry are deeply related as shown by the Nielsen-Ninomiya no-go theorem: the following desirable properties of a massless free lattice Dirac operator $D(x)$

  1. $D(x)$ is local (bounded by $Ce^{-|x|/\rho}$ with $\rho \propto a$);
  2. $\tilde{D}(p) = i\gamma_\mu p_\mu + O(ap^2)$ for $p \ll \pi/a$;
  3. $\tilde{D}(p)$ is invertible for $p \neq 0$ (no massless doublers);
  4. $\gamma_5 D + D\gamma_5 = 0$ (chiral symmetry);

  can not hold simultaneously.

- Since it is not possible to give up the first three properties, it seems impossible to have chiral symmetry on the lattice. Therefore, this last problem seems not to be a peculiarity of Wilson fermions.

- Way out: instead of property 4, require the massless Dirac operator D to satisfy the Ginsparg-Wilson relation $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$.

- $\Rightarrow$ the action $\int d^4x\bar{\psi}D\psi$ is invariant under the modified chiral symmetry:

$$\psi \to \psi + \epsilon\hat{\gamma}_5\psi, \qquad \bar{\psi} \to \bar{\psi} + \epsilon\bar{\psi}\gamma_5, \qquad \text{where} \qquad \hat{\gamma}_5 = \gamma_5(1 - aD)$$

- The chiral projectors $\hat{P}_{\mp} = \frac{1}{2}(1 \mp \hat{\gamma}_5)$ and $P_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$ (for fermion and antifermion fields respectively) allow to eliminate the right-handed (left-handed) components by imposing $\hat{P}_{\mp}\psi = \psi$ and $\bar{\psi}P_{\pm} = \bar{\psi}$.

- An operator which satisfy the Ginsparg-Wilson relation is given by

$$D = \frac{1}{a}\Big\{1 - \frac{(1-aD_w)}{\sqrt{(1-aD_w)^\dagger(1-aD_w)}}\Big\}$$

which coincides with D_w up to $O(a)$ lattice artifacts.

- Despite the exact chiral symmetry of the action, the fermion meausure is not invariant und the axial anomaly is recovered *à la* Fujikawa.

- Operator mixing is now constrained by chiral symmetry exactly as in the continuum (e.g. there is no additive mass renormalization).

- Chiral gauge theories can be meaningfully defined on the lattice. There is a proof in PT and a non-perturbative proof for abelian gauge groups.

- Despite this success, numerical simulations are still much more expensive than for Wilson fermions. We therefore continue the discussion of the latter.

Chiral Ward-Takahashi Identities on the Lattice

- Despite the explicit breaking of chiral symmetry with Wilson fermions, it is still possible to construct a set of currents that, for $a \to 0$ are partially conserved and obey Current Algebra $\Rightarrow$ it is also possible to construct operators with well defined chiral transformations properties.

- Lattice analog of continuum WTI. Local infinitesimal $SU_L(N_f) \times SU_R(N_f)$

non-anomalous chiral transformation:

$$\delta\psi(x) = \epsilon \left[\alpha_V^a(x)\frac{\lambda^a}{2} + \alpha_A^a(x)\frac{\lambda^a}{2}\gamma_5\right]\psi(x)$$
$$\delta\bar{\psi}(x) = -\epsilon\bar{\psi}(x)\left[\alpha_V^a(x)\frac{\lambda^a}{2} - \alpha_A^a(x)\frac{\lambda^a}{2}\gamma_5\right] \tag{1}$$

- the time-ordered vacuum expectation value of a multilocal operator $O(x_1, ..., x_n) \equiv O_1(x_1)...O_n(x_n)$ is given by the lattice functional integral

$$\langle O(x_1,\ldots,x_n)\rangle = \frac{1}{Z}\int [dU\,d\psi\,d\bar{\psi}]O(x_1,\ldots,x_n)e^{-S^W_{QCD}[U,\bar{\psi},\psi]} \tag{2}$$

- WTI are a consequence of the invariance of (2) under local changes of the fermionic integration variables. Under (1) we have

$$\left\langle \left.\frac{\delta O(x_1,\ldots,x_n)}{\delta\alpha^a(x)}\right|_{\alpha^a(x)=0}\right\rangle - \left\langle O(x_1,\ldots,x_n)\left.\frac{\delta S_{QCD}}{\delta\alpha^a(x)}\right|_{\alpha^a(x)=0}\right\rangle = 0 \tag{3}$$

where we consider either $\alpha^a = \alpha_V^a \neq 0$, $\alpha_A^a = 0$ (vector transformation) or $\alpha^a = \alpha_A^a \neq 0$, $\alpha_V^a = 0$ (axial transformation).

Vector WT Identity on the Lattice

$$i\langle\frac{\delta O(x_1,\ldots,x_n)}{\delta\alpha_V^a(x)}\rangle = \langle O(x_1,\ldots,x_n)\nabla_x^\mu \widetilde{V}_\mu^a(x)\rangle + \\ +\langle O(x_1,\ldots,x_n)\bar{\psi}(x)[\frac{\lambda^a}{2},m]\psi(x)\ ,\rangle$$

where $(x_1 \neq x_2 \neq \ldots x_n)$, $\nabla_x^\mu f(x) = (f(x) - f(x-\mu))/a$ and

$$\widetilde{V}_\mu^a(x) = \frac{1}{2}[\bar{\psi}(x)(\gamma_\mu - 1)U_\mu(x)\frac{\lambda^a}{2}\psi(x+\mu) + \\ +\bar{\psi}(x+\mu)(\gamma_\mu+1)U_\mu^\dagger(x)\frac{\lambda^a}{2}\psi(x)]$$

- In the following we are interested in on-shell matrix elements of the currents between hadronic states (created by $O(x_1,\ldots,x_n)$). Contact terms coming from the l.h.s of (3) are irrelevant in the large distance limit in which the on-shell matrix elements are extracted. Then we see that the point-split vector current is partially conserved (PCVC)

$$\langle\alpha|\nabla_x^\mu \widetilde{V}_\mu^a(x)|\beta\rangle = \langle\alpha|\bar{\psi}(x)[\frac{\lambda^a}{2},m]\psi(x)|\beta\rangle$$

- In the limit of degenerate bare quark masses it becomes $\langle\alpha|\nabla_x^\mu \widetilde{V}_\mu^a|\beta\rangle = 0$.

- By imposing the analogous WTI of the renormalized theory it is possible to show that $Z_{\widetilde{V}} = 1$ (*non-renormalization theorem*). This result is valid for any non-anomalous PCC for which the global symmetry is bronken by a term of dimension lower than the dimension of the lagrangian.

- The local vector current $V_\mu^a(x) = \bar{\psi}(x)\frac{\lambda^a}{2}\gamma_\mu\psi(x)$ is not conserved on the lattice but it differ from the conserved current by a finite renormalization.

- relation between the conserved current and the local current

$$\begin{aligned}\widetilde{V}_\mu^a(x) = V_\mu^a(x) \quad &+ \quad \tfrac{1}{2}\{\bar{\psi}(x)(\gamma_\mu - 1)\tfrac{\lambda^a}{2}[U_\mu(x)\psi(x+\mu) - \psi(x)] \\ &+ \quad [\bar{\psi}(x+\mu)U_\mu^\dagger(x) - \bar{\psi}(x)](\gamma_\mu + 1)\tfrac{\lambda^a}{2}\psi(x)\}\end{aligned}$$

where the second term on the r.h.s. is a 4-dim operator Δ_μ^a times a.

- Consider the amputated Green's function $\Lambda_{\widetilde{V}}(p)$ defined by:

$$\Lambda_O(p_1,p_2) = \tilde{S}^{-1}(p_1)\tilde{G}_O(p_1,p_2)\tilde{S}^{-1}(p_2), \qquad \tilde{S}(p) = a^4\textstyle\sum_{x_1} e^{ipx_1}\langle\psi(x_1)\bar{\psi}(x_2)\rangle,$$

$$\tilde{G}_O(p_1,p_2) = a^8\textstyle\sum_{x_1,x_2} e^{i(p_1x_1 - p_2x_2)}\langle\psi(x_1)O(0)\bar{\psi}(x_2)\rangle$$

$\Rightarrow \Lambda_{\widetilde{V}}(p) = \Lambda_V(p) + a\Lambda_\Delta(p)$. $a\Lambda_\Delta(p)$ vanishes at tree-level as $a \to 0$.

- Beyond tree-level however it contributes due to power divergences induced by mixing with lower dimensional operators. Mixing with operators of the same dimension gives at most logarithmic terms which vanish, when $a \to 0$, as $a \ln(ap)$. The only lower dimensional operator is V_μ^a $\Rightarrow$ one has a^{-1} divergences (without logs) multiplied by the a factor in front $\Rightarrow a\Lambda_\Delta(p)$ gives finite contributions which combine with those from $\Lambda_V(p)$ to give $Z_{\widetilde{V}} = 1$. $\Rightarrow Z_V(g^2) \neq 1$ (being finite it can only depend on the coupling g^2, not on μ).

- The WTI can now be expreessed in terms of V_μ^a, it suffices to substitute $\tilde{V}_\mu^a$ by $Z_V V_\mu^a$. Z_V can be computed non-perturbatively from suitable WTI.

Axial WT Identity on the Lattice

- In the continuum, out of the chiral limit, the non-singlet axial current is partially conserved (PCAC) $\partial_\mu(\bar{\psi}(x)\gamma_\mu\gamma_5\frac{\lambda^a}{2}\psi(x)) = \bar{\psi}(x)\{\frac{\lambda^a}{2}, m\}\psi(x)$ and $Z_A = 1$ for the non-renormalization theorem.

- On the lattice, however, the (5 dimensional) Wilson term breaks explicitly chiral symmetry $\Rightarrow$ As we will see in a moment, Z_A remains finite but $\neq 1$.

$$
\begin{aligned}
i\langle\tfrac{\delta O(x_1,\ldots,x_n)}{\delta\alpha_A^a(x)}\rangle &= \langle O(x_1,\ldots,x_n)\nabla_x^\mu \widetilde{A}_\mu^a(x)\rangle \\
&\quad -\langle O(x_1,\ldots,x_n)\bar{\psi}(x)\{\tfrac{\lambda^a}{2},M_0\}\gamma_5\psi(x)\rangle - \langle O(x_1,\ldots,x_n)X^a(x)\rangle
\end{aligned}
$$

where
$$
\begin{aligned}
\widetilde{A}_\mu^a(x) &= \tfrac{1}{2}[\bar{\psi}(x)\gamma_\mu\gamma_5 U_\mu(x)\tfrac{\lambda^a}{2}\psi(x+\mu) \\
&+ \bar{\psi}(x+\mu)\gamma_\mu\gamma_5 U_\mu^\dagger(x)\tfrac{\lambda^a}{2}\psi(x)]
\end{aligned}
$$

X^a is the variation of the Wilson term under the axial transformation. It's a 5-dim operator multiplied by a which can not be cast in the form of a 4-divergence $\Rightarrow$ vanishes at tree-level when $a \to 0$. It has been shown that it has divergent matrix elements beyond tree-level.

- It is possible to suitably redefine operators and bare parameters in such a way that the continuum renormalized axial WTI has the same form it assumes when chiral symmetry is preserved.

- One can define $\overline{X}^a$ which is multiplicatively renormalizable and vanishes as $a \to 0$: subtract from X^a the operators of lower dimensionality (allowed by symmetries) with whom it mixes

$$\overline{X}^a(x) = X^a(x) + \bar{\psi}(x)\{\tfrac{\lambda^a}{2}, \overline{m}\}\gamma_5\psi(x) + (Z_{\widetilde{A}} - 1)\nabla_x^\mu \widetilde{A}_\mu^a(x)$$

- dimensional analysis + existence of continuum limit $\Rightarrow Z_{\widetilde{A}}(g^2, am)$ is finite, whereas $\overline{m}(g,m)$ diverges linearly as a^{-1} (without logarithmic divergences).
- insertion of $\overline{X}^a(x)$ with elementary fields vanishes as $a \ln ap$ when $a \to 0$ while extra divergent localized contributions (δsor derivatives of δs) appears when $\overline{X}^a(x)$ is inserted with composite operators $\Rightarrow$

$$Z_{\widetilde{A}}\langle\alpha|\nabla_x^\mu \widetilde{A}_\mu^a(x)|\beta\rangle = \langle\alpha|\bar{\psi}\{\tfrac{\lambda^a}{2}, (m - \overline{m})\}\gamma_5\psi|\beta\rangle + \langle\alpha|\overline{X}^a|\beta\rangle$$

- $\langle\alpha|\overline{X}^a|\beta\rangle \to 0$ when $a \to 0$ and we recover the standard PCAC provided the lattice renormalized axial current to be defined as $\hat{A}_\mu^a \equiv Z_{\tilde{A}}\tilde{A}_\mu^a$ and we identify the chiral limit as the one in which $m = \overline{m}(g^2, m)$, whose solution is called m_{cr}. It easy to prove that m_{cr} has to be flavour singlet and that $Z_{\tilde{A}}$, analogously to Z_V, can only depend on g^2.
- The separation between $\nabla_\mu \tilde{A}_\mu$ and X^a in is not unique. We can always add X^a a total divergence and correspondingly modify the definition of A_μ. Z_A changes in such a way to preserve the PCAC relation $\hat{A}_\mu^a \equiv Z_{\tilde{A}}\tilde{A}_\mu^a \equiv Z_A A_\mu^a$. One can for example use the local axial current $A_\mu^a \equiv \bar{\psi}(x)\frac{\lambda^a}{2}\gamma_\mu\gamma_5\psi(x)$.

- Non-perturbatively, the vanishing of the matrix elements of $\overline{X}^a$ between on-shell hadron states may determine only the ratio $\rho = Z_A^{-1}(m - \overline{m})$ and another condition is needed to compute separately Z_A and $(m - \overline{m})$.

Chiral Composite Operators

- Let $O^i_{[n]}$ be a basis of operators which at tree-level transform according to the irreducible representation $[n]$ of the chiral group

$$\frac{1}{i}\frac{\delta O^i_{[n]}(0)}{\delta \alpha^f} = (r^f_{[n]})^{ij} O^j_{[n]}(0)$$

with $r^f_{[n]}$ the f^{th}generator of an axial transformation in the representation $[n]$.

- Wilson term $\Rightarrow$ radiative corrections induce mixing among operators with different ($nominal$) chiral properties.

- Is it possible to find suitable linear combinations (with coefficients $c^{ij}_{[n,n']}$) of operators belonging to different chiral representations which, up to $O(a)$, will trasform according to the irrep $[n]$? $\exists$ $\{c^{ij}_{[n,n']}\}$ such that

$$\hat{O}^i_{[n]} = Z_{O^i}(O^i_{[n]} + \sum_{n' \neq n, j} c^{ij}_{[n,n']} O^j_{[n']}) \equiv Z_{O^i} \tilde{O}^i_{[n]}$$

obeys WTIs formally identical to the continuum ones (for simplicity we consider operators $O^i_{[n]}$ multiplicatively renormalizable in the continuum)?

- Important remark: with a generalization of the argument proposed for Z_V one can show that WTIs can only determine scale independent (i.e. finite) mixing coefficients (e.g. the c's), RCs (e.g. Z_A, Z_V) or ratios of RCs for which the dependence on the renormalization scale cancel out (e.g. Z_P/Z_S, Z_{O^i}/Z_{O^j}).

- To fix the c's and the ratios Z_{O^i}/Z_{O^j}, we write the renormalized integrated lattice axial WTI

$$\sum_x \nabla_\mu \langle h_1 | T(\hat{A}^f_\mu(x) \hat{O}^i_{[n]}(0)) \, | h_2 \rangle =$$

$$= \sum_x \Big[\langle h_1 | T(\bar{\psi} \gamma_5 \{ \tfrac{\lambda^f}{2}, m - \overline{m} \} \psi(x) \hat{O}^i_{[n]}(0)) \, | h_2 \rangle +$$

$$+ \langle h_1 | T(\overline{X}^f(x) \hat{O}^i_{[n]}(0)) \, | h_2 \rangle \Big] - i \langle h_1 | \frac{\delta \hat{O}^i_{[n]}(0)}{\delta \alpha^f} \, | h_2 \rangle$$

where we have neglected the contribution of the axial rotation of O_{h_1} and O_{h_2} (which create the hadrons h_1 and h_2 from the vacuum). These terms are localized at large positive and negative times t_1 and t_2 and can be safely neglected at least if the hadrons h_1 and h_2 are lighter than the corresponding chiral rotated states.

- In the chiral limit, $m = m_{\rm cr}$ and the first tirm on the r.h.s. is missing but the presence of Goldstone bosons gives a non vanishing contribution to the l.h.s. Out of the chiral limit there are no Goldstone bosons and therefore the l.h.s. vanishes but not the first term of the r.h.s.

- Thus, in the chiral limit we can compute c's and Z_{O^i}/Z_{O^j} from the condition

$$\sum_x \langle h_1|T(\overline{X}^f(x)\hat{O}^i_{[n]}(0))\,|h_2\rangle - i\langle h_1|\frac{\delta\hat{O}^i_{[n]}(0)}{\delta\alpha^f}\,|h_2\rangle =$$
$$= (r^f_{[n]})^{ij}\langle h_1|\hat{O}^j_{[n]}(0)\,|h_2\rangle$$

- The insertion of $\overline{X}$ vanishes on-shell but it gives raise to localized contact terms when it touches $\hat{O}$. The equations which fix c's and Z_{O^i}/Z_{O^j} express the fact that the contact terms combine with the messy contribution from the second term in the l.h.s. to give for the $\hat{O}^i_{[n]}$ the continuum WTI

$$\sum_x \nabla_\mu \langle h_1 | T(\hat{A}^f_\mu(x) \hat{O}^i_{[n]}(0)) \, | h_2 \rangle \equiv \langle h_1 | [Q^f_5, \hat{O}^i_{[n]}(0)] | h_2 \rangle = (r^f_{[n]})^{ij} \langle h_1 | \hat{O}^j_{[n]}(0) \, | h_2 \rangle$$

where $Q^f_5 \equiv \sum_{\mathbf{x}} \hat{A}^f_0(\mathbf{x}, t)$ are the axial charges.

- The same results can be found out of the chiral limit. In this case however there are subtleties related to the presence of extra power divergences which arises because of the insertion of the (integrated) pseudoscalar density in correlators containing $\hat{O}_{[n]}$.

- $\tilde{O}$ defined by this procedure has the same renormalization properties of the continuum one (for simplicity we consider a multiplicatively renormalizable operator). The overall scale-dependent (and thus logarithmically divergent) renormalization constant Z_O is needed to obtain the renormalized operator $\hat{O}(\mu) = Z_O(\mu a, \Lambda_{QCD} a) \tilde{O}(a)$ (we have assumed a mass-independent renormalizationn scheme which is guaranteed by computing the RCs in the chiral limit). Only at this point we can perform the continuum limit of the matrix element computed on the lattice.

- Crucial observation: the subtraction of lower dimensional operators (multiplied by power-divergent mixing coefficients) must be performed non-perturbatively, because non-perturbative contributions of the form $\propto \exp(-1/2\beta_0 g^2)$, when multiplied by a^{-1}, will lead, as $a \to 0$, to a non-vanishing constant contribution $a^{-1}\exp(-1/2\beta_0 g^2) \sim \Lambda_{QCD}$.

- For the logarithmically divergent RCs and even the finite mixing coefficient/RCs, it turns out that bare lattice PT is badly convergent. Various recipes have been tried out in order to improve the convergence of the perturbative expansion, however none of them seems completely reliable and univerally applicable (without considering the fact that lattice perturbation theory can be hardly pushed beyond one-loop). There are several regularizations where some of the bilinears have RCs of the order of $\sim 0.4 \div 0.5$ and it is difficult to trust a parturbative calculation at one-loop which gives a result so different from 1. For these reasons non-perturbative renormalization has been developed and intensively studied in the last years.

- Two types of scheme have been developed in order to compute non-perturbatively scale-dependent RCs: infinite-volume schemes (the RI-MOM scheme) and finite-volume schemes (the Schrödinger functional scheme). They will be presented later in these lectures.

- Concerning the actual determination of the mixing coefficients c's in Monte Carlo simulations, we notice that, by varying the external hadronic states, one can get, in principle, a number of independent conditions sufficient to fix them completely. However, this is unpractical, because it would require high precision Monte Carlo measurements of a large number of hadronic matrix elements. Various possible strategies have been proposed to overcome these difficulties, some of them being presented later in these lectures.

- A final observation concerns the case in which one is interested in $\langle h_1 | i\frac{\delta \hat{O}_{[n]}}{\delta \alpha^f} | h_2 \rangle$ where $i\frac{\delta \hat{O}_{[n]}}{\delta \alpha^f}$ presents spurious lattice mixing while $\hat{O}_{[n]}$, thanks to additional symmetries, renormalizes as in the continuum (e.g. multiplicatively). Then one may use the renormalized WTI above to obtain directly $\langle h_1 | i\frac{\delta \hat{O}_{[n]}}{\delta \alpha^f} | h_2 \rangle$ by computing the matrix element of the "simpler" operator $\hat{O}_{[n]}$ together with the integrated divergence of the axial current. This matrix element does not present in fact spurious lattice mixing! [Becirevic,..., Papinutto (2000)]

Non-perturbative Renormalization via WTI on Hadron States

- $\langle \alpha | \overline{X}^a | \beta \rangle \to 0$ as $a \to 0$ determines only $\rho = Z_A^{-1}(m - \overline{m})$.
- ρ extracted from the axial WTI with $O(x_1) = P^{21}(x_1) = \bar{\psi}_2(x_1)\gamma_5\psi_1(x_1)$:

$$Z_A \nabla_x^\mu \langle A_\mu^{12}(x) P^{21}(x_1) \rangle =$$
$$= \langle \overline{X}^{12}(x) P^{21}(x_1) \rangle \; + \; \left[m_1 + m_2 - \overline{m}_1 - \overline{m}_2 \right] \langle P^{12}(x) P^{21}(x_1) \rangle$$

where $\overline{m}_i(g^2, m)$ and m_i are the i^{th} diagonal element of $\overline{m}(g^2, m)$ and m.

- The renormalized quark mass is defined as (noticing that $\overline{m}(m_{\rm cr}) = m_{\rm cr}$)

$$\hat{m} = \bar{Z}_m [\, m - \overline{m}(m) \,] = \bar{Z}_m [\, m - m_{\rm cr} \; - \; \frac{\partial \overline{m}}{\partial m}\Big|_{m_{\rm cr}} (m - m_{\rm cr}) + \ldots]$$

- the renormalized axial WTI $\Rightarrow$ $Z_P = 1/\bar{Z}_m$. Since $\langle \overline{X}(x) \hat{P}(x_1) \rangle \rightarrow 0$ when $a \rightarrow 0$

$$2\rho^{12} = Z_A^{-1} [m_1 + m_2 - \overline{m}_1 - \overline{m}_2] = \frac{\nabla_\mu^x \langle A_\mu^{12}(x) P^{21}(x_1) \rangle}{\langle P^{12}(x) P^{21}(x_1) \rangle} = \frac{\nabla_0^{x_0} \int d\mathbf{x} \langle A_0^{12}(x) P^{21}(x_1) \rangle}{\int d\mathbf{x} \langle P^{12}(x) P^{21}(x_1) \rangle}$$

- In order to determine Z_A, Z_V and Z_P/Z_S we need to impose other conditions. The idea is to impose that the non-linear relations of Current Algebra should be satisfied by the renormalized currents $V_\mu^a = Z_V V_\mu^a$, $\hat{A}_\mu^a = Z_A A_\mu^a$.

- take the axial WTI with $O(x_1, x_2) = A^b_\nu(x_1) V^c_\rho(x_2)$:

$$\begin{aligned}
\nabla_\mu \langle \hat{A}^a_\mu(x) A^b_\nu(x_1) V^c_\rho(x_2) \rangle \; = \\
&= \langle \bar{\psi} \gamma_5 \{ \tfrac{\lambda^a}{2}, m - \overline{m} \} \psi(x) A^b_\nu(x_1) V^c_\rho(x_2) \rangle \; + \; \langle \overline{X}^a(x) A^b_\nu(x_1) V^c_\rho(x_2) \rangle \\
&- \, i f^{abd} \delta(x - x_1) \langle V^d_\nu(x_1) V^c_\rho(x_2) \rangle \; - \; i f^{acd} \delta(x) \langle A^b_\nu(x_1) A^d_\rho(x_2) \rangle
\end{aligned}$$

- The insertion of $\overline{X}^a$ with $A^b_\nu(x_1) V^c_\rho(x_2)$ is a sum of terms localized at $x = x_1, x_2$. Using flavor symmetry one has

$$\begin{aligned}
\langle \overline{X}^a(x) A^b_\nu(x_1) V^c_\rho(x_2) \rangle = & \; - i k_1 f^{abd} \delta(x - x_1) \langle V^d_\nu(x_1) V^c_\rho(x_2) \rangle \\
& - i k_2 f^{acd} \delta(x - x_2) \langle A^b_\nu(x_1) A^d_\rho(x_2) \rangle \; + \; \ldots
\end{aligned}$$

where $\ldots$ represent localized (Schwinger) terms which vanish after integration upon x. The axial WTI should have the same form as the continuum one $\Rightarrow$

$$k_1 = \frac{Z_V}{Z_A} - 1 \qquad\qquad k_2 = \frac{Z_A}{Z_V} - 1$$

$$\langle[\nabla^\mu_x A^a_\mu(x) - \bar\psi(x)\{\tfrac{\lambda^a}{2},\rho\}\gamma_5\psi(x)]A^b_\nu(x_1)V^c_\rho(x_2)\rangle =$$
$$+i\frac{Z_V}{Z_A^2}f^{abd}\delta(x-x_1)\langle V^d_\nu(x_1)V^c_\rho(x_2)\rangle + i\frac{1}{Z_V}f^{acd}\delta(x-x_2)\langle A^b_\nu(x_1)A^d_\rho(x_2)\rangle$$

- Out of the chiral limit, after integration over x and over $\mathbf{x}_1$ (with $x_1^0 \neq x_2^0$ to eliminate Schwinger terms) we obtain

$$\int dx d\mathbf{x}_1 \langle\bar\psi(x)\{\tfrac{1}{2}\lambda^a,\rho\}\gamma_5\psi(x)A^b_\nu(x_1)V^c_\rho(x_2)\rangle =$$
$$-i\frac{Z_V}{Z_A^2}f^{abd}\int d\mathbf{x}_1\langle V^d_\nu(x_1)V^c_\rho(x_2)\rangle - i\frac{1}{Z_V}f^{acd}\int d\mathbf{x}_1\langle A^b_\nu(x_1)A^d_\rho(x_2)\rangle$$

- Taking $\nu=\rho=0$, the first term on the r.h.s. is zero (conserved vector charge on the vacuum) $\Rightarrow$ determine Z_V by knowing ρ.

- Taking $\nu=\rho=k$ (spatial) $\Rightarrow$ determine Z_A by knowing ρ and Z_V.

- Taking $O(x_1,x_2) = P^{12}(x_1)P^{31}(x_2)$, where $P^{12} = \bar\psi_1\gamma_5\psi_2$ and $x \neq x_1, x_2$, so to avoid contact terms, the vector WTI

$$Z_V \nabla^x_\mu \langle P^{12}(x_1)V^{23}_\mu(x)P^{31}(x_2)\rangle = (m_2-m_3)\langle P^{12}(x_1)S^{23}(x)P^{31}(x_2)\rangle$$

- The renormalized mass can be defined to be $\hat{m} = Z_m[m - m_{\rm cr}]$ where the chiral limit is then $m \to m_{\rm cr}$. In PT, at tree-level, $m_{\rm cr} = -4/a$. By requiring that the renormalized quantities obey the nominal vector WI $\Rightarrow Z_S = Z_m^{-1}$

- Analogously, when $O(x_1, x_2) = S^g(x_1)P^h(x_2)$ (with $f \neq g, h$)

$$\int d^4x \int d\mathbf{x}_1 \; \langle \bar{\psi}(x)\{\tfrac{1}{2}\lambda^f, \rho\}\psi(x) S^g(x_1) P^h(x_2)\rangle =$$

$$\tfrac{Z_P}{Z_A Z_S} d^{fgl} \int d^3\mathbf{x}_1 \; \langle P^l(x_1) P^h(x_2)\rangle + \tfrac{Z_S}{Z_A Z_P} d^{fhl} \int d^3\mathbf{x}_1 \; \langle S^g(x_1) S^l(x_2)\rangle$$

  one can extract Z_P/Z_S which is of course a function of g^2 only $\Rightarrow$ P and S have the same anomalous dimension.

- Z_P/Z_S can be obtained also from the two ways of defining the renormalized quark mass $\hat{m}$

$$\tfrac{Z_P}{Z_S} = \tfrac{m - \overline{m}(m)}{m - m_{\rm cr}} = 1 - \tfrac{\partial \overline{m}(g^2, m)}{\partial m}\Big|_{m = m_{\rm cr}} + \dots$$

- In practice one obtain Z_P/Z_S by computing the slope of

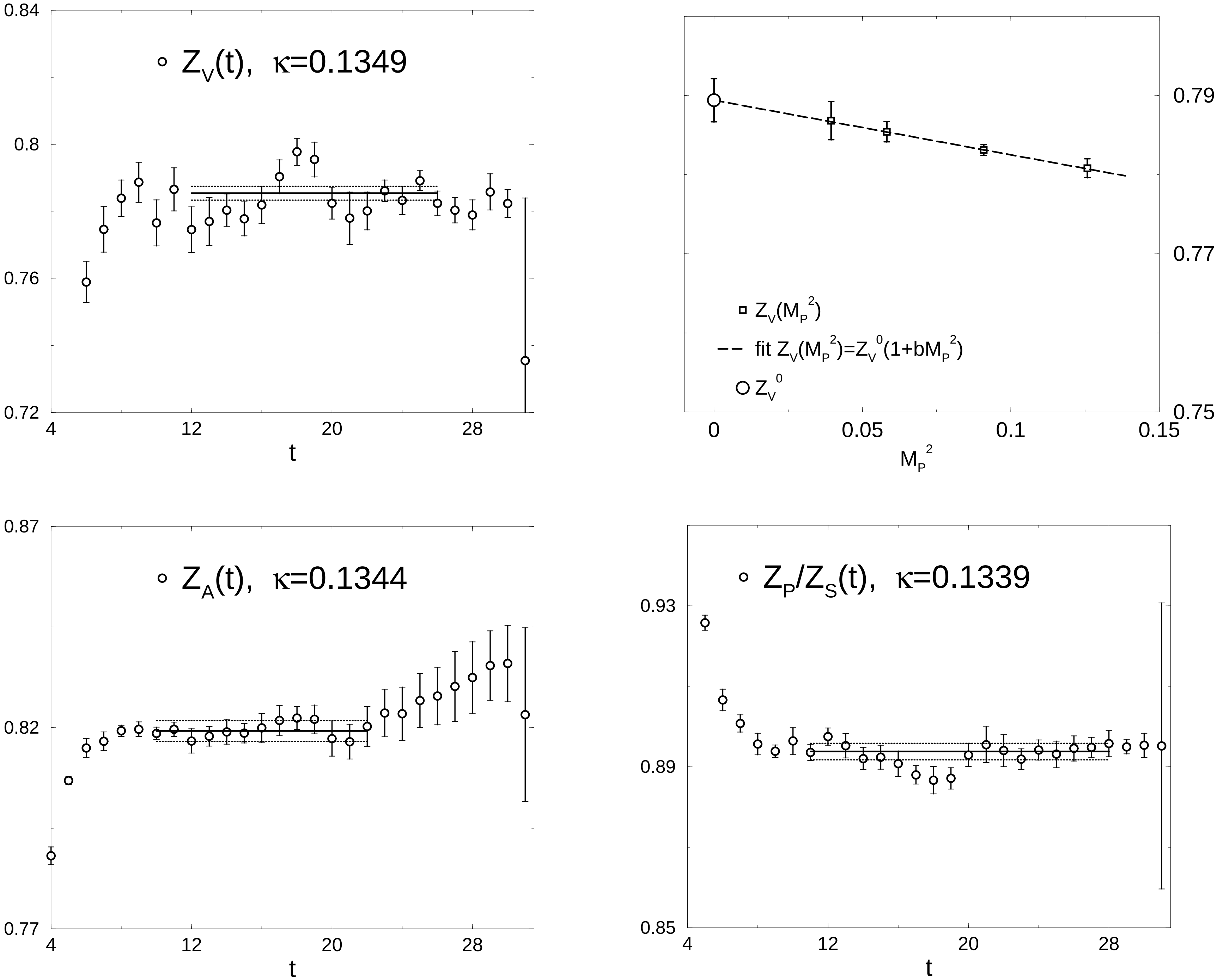
$$\rho Z_A = m - \overline{m} = \frac{Z_P}{Z_S}\left[m - m_{\mathrm{cr}}\right]$$

as function of m (it depend on Z_A but not on m_{cr}).

- In the unimproved theory all this results are valid up to $O(a)$ terms. In the $O(a)$ improved theory, results are valid up to $O(a^2)$ terms.

- Numerical results from [Becirevic, Gimenez, Lubicz, Martinelli, Papinutto, Reyes (2005)]

		RI/MOM	WTI		SF	BPT 1-loop	
	β	ROME	ROME	LANL	ALPHA	$c_{SW}=1$	c_{SW}^{NP}
Z_V	6.2	0.783(3)	0.789(2)	0.7874(4)	0.792(1)	0.7959	0.8463
	6.4	0.801(2)	0.804(2)	0.8018(5)	0.803(1)	0.8076	0.8480
Z_A	6.2	0.819(3)	0.812(5)(2)	0.818(5)	0.807(8)	0.8163	0.8624
	6.4	0.832(3)	0.843(10)(1)	0.827(4)	0.827(8)	0.8269	0.8628
$\frac{Z_P}{Z_S}$	6.2	0.877(5)	0.877(5)(1)	0.884(3)	[0.886(9)]	0.9449	0.9545
	6.4	0.894(3)	0.914(10)(1)	0.901(5)	[0.908(9)]	0.9491	0.9594



Z_V(t), κ=0.1349
0.84
0.8
0.76
0.72
4
12
20
28
t
Z_V(M_P^2)
fit Z_V(M_P^2)=Z_V^0(1+bM_P^2)
Z_V^0
0.79
0.77
0.75
0
0.05
0.1
0.15
M_P^2
Z_A(t), κ=0.1344
0.87
0.82
0.77
Z_P/Z_S(t), κ=0.1339
0.93
0.89
0.85