Lattice QCD and Non-perturbative Renormalization

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Lecture 4: Symanzik Improvement

Local Continuum Effective Theory

- Close to the continuum limit the lattice QCD may be described in terms of a local effective theory (LET) with action

$$S_{\mathrm{eff}} = \int \mathrm{d}^4x \left\{ \mathcal{L}_0(x) + a\mathcal{L}_1(x) + a^2\mathcal{L}_2(x) + \ldots \right\}$$

  where $\mathcal{L}_0$ denotes the continuum QCD lagrangian while the other terms are to be interpreted as operator insertions in the continuum theory.

- Originally, Symanzik defines the continuum theory using dimensional regularization, but we could also employ a lattice with spacing very much smaller then a to give a precise meaning to $\mathcal{L}_0(x)$ and the operator insertions.

- $\mathcal{L}_k$'s, $k \geq 1$, are linear combinations of local composite operators of dimension $4 + k$ which respect the symmetries of the lattice theory. The dimension counting here includes the (non-negative) powers of the quark mass m by which some of the fields may be multiplied.

- $\mathcal{L}_1$ for instance must be a linear combination of

$$\mathcal{O}_1 = \bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi \quad \mathcal{O}_3 = m\mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu}\} \quad \mathcal{O}_5 = m^2\,\bar{\psi}\psi$$
$$\mathcal{O}_2 = \bar{\psi}D_\mu D_\mu\psi + \bar{\psi}\overleftarrow{D}_\mu\overleftarrow{D}_\mu\psi \quad \mathcal{O}_4 = m\{\bar{\psi}\gamma_\mu D_\mu\psi + \bar{\psi}\overleftarrow{D}_\mu\gamma_\mu\psi\}$$

- Cut-off effects originate not only from the lattice action but also from the local composite fields that one is interested in. Let $\phi(x)$ be some local gauge invariant field constructed from the quark and gluon fields on the lattice (for simplicity we consider $\phi(x)$ to be multiplicatively renormalizable). We expect the connected renormalized n-point correlation functions

$$G_n(x_1,\ldots,x_n) = (Z_\phi)^n\,\langle\phi(x_1)\ldots\phi(x_n)\rangle_{\mathrm{con}}$$

  to have a well-defined continuum limit if all points $x_1,\ldots,x_n$ are kept at non-zero distances from one another.

- In the LET the renormalized lattice field $Z_\phi\phi(x)$ is represented by an effective field
$$\phi_{\mathrm{eff}}(x) = \phi_0(x) + a\phi_1(x) + a^2\phi_2(x) + \ldots$$

where the fields $\phi_k(x)$ are linear combinations of local fields with the appropriate dimension and symmetry properties.

- To order a the lattice correlation functions are then given by

$$\begin{aligned} G_n(x_1,\ldots,x_n) &= \langle\phi_0(x_1)\ldots\phi_0(x_n)\rangle_{\rm con} \\ &\quad -a\int \mathrm{d}^4y\ \langle\phi_0(x_1)\ldots\phi_0(x_n)\mathcal{L}_1(y)\rangle_{\rm con} \\ &\quad +a\sum_{k=1}^{n}\langle\phi_0(x_1)\ldots\phi_1(x_k)\ldots\phi_0(x_n)\rangle_{\rm con}+O(a^2), \end{aligned}$$

where the expectation values on the right-hand side are to be taken in the continuum theory with lagrangian $\mathcal{L}_0$.

- The second term is the contribution of the $O(a)$ correction in the effective action. The integral over y in general diverges at the points $y=x_k$. A subtraction prescription is needed but its precise definition is unimportant because the arbitrariness that one has amounts to a local operator insertion at these points, i.e. to a redefinition of the field $\phi_1(x)$.

- Not all the a-dependence comes from the explicit factors of a. Other sources are the operators ϕ_1 and $\mathcal{L}_1$, which are linear combinations of some basis of fields. While the basis elements are a-independent, the coefficients are not (although they vary slowly with a). In PT they are polynomials in $\ln a$.

- Remark: all on-shell quantities can be extracted from correlation functions of local composite fields and these correlation functions are only required at non-zero physical distances $\Rightarrow$ the LET can be simplified considerably if attention is restricted to such correlation functions.

- Let's consider the term containing $\mathcal{L}_1(y)$. $\mathcal{L}_1(y)$ is a linear combination of the fields displayed above. As long as y is kept away from x_k, we can apply the field equations of the continuum theory to conclude that certain linear combinations of these fields do not contribute. This remains true after integration over y up to contact terms that arise in $x_1, \ldots, x_k$ and which amount to operator insertions. Symmetries allow only contact terms with the same form of ϕ_1 and thus they can be compensated by a redefinition of ϕ_1

- The linear combinations of $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5\}$ which vanish can be found at tree-level in PT by applying the classical field equations. We find two relations which allow eliminate $\mathcal{O}_2$ and $\mathcal{O}_4$. At non-zero couplings the coefficients in the linear combinations change but the linear dependence still hold (barring dingular events). We may take $\mathcal{L}_1$ to be a linear combination of $\mathcal{O}_1$,$\mathcal{O}_3$, $\mathcal{O}_5$.

- similiar arguments may be used to eliminate some of the basis fields from which ϕ_1 is constructed. Since $x_1, \ldots, x_k$ are kept at non-zero distances no contact terms can arise when the field equations are applied.

Improved Lattice Action

- The orginal idea of improvement consists in adding, to both the action and operators, a complete set of higher-dimensional (*irrelevant*) operators, the coefficients of which are tuned as to cancel finite cut-off effects (to a desired order of a).

- We start with the lattice action. We add a suitable $O(a)$ counterterm to the Wilson action such that the $O(a)$ term in the LET is cancelled. From what we have said above, $\mathcal{L}_1(y)$ can be made to vanish by adding a counterterm

$$a^5 \sum_x \left\{ c_1 \hat{\mathcal{O}}_1 + c_3 \hat{\mathcal{O}}_3 + c_5 \hat{\mathcal{O}}_5 \right\}$$

where $\hat{\mathcal{O}}_k$ is some lattice representation of $\mathcal{O}_k$.

- Apart from the renormalization of the bare parameters and the tuning of the c_k's, the discretization ambiguities that one has here are of $O(a^2)$.

- We can choose to represent $\mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu}\}$ and $\bar{\psi}\psi$ by the Wilson plaquette and the local scalar density already appearing in the Wilson action. The $O(a)$ counterterms $c_3\hat{\mathcal{O}}_3$ and $c_5\hat{\mathcal{O}}_5$ then amount to a renormalization of g and m (this will not be insignificant as explained in the following).

- For the on-shell $O(a)$ improved action we thus obtain $S_{\mathrm{impr}}[U,\bar{\psi},\psi] = S_W[U,\bar{\psi},\psi] + \delta S[U,\bar{\psi},\psi]$ where S_W is the Wilson action and

$$\delta S[U\bar{\psi},\psi] = a^5 c_{\mathrm{sw}} \tfrac{i}{4} \sum_x \bar{\psi}(x)\sigma_{\mu\nu}\hat{F}_{\mu\nu}(x)\psi(x)$$

  where $\hat{F}_{\mu\nu}$ is a lattice representation of $F_{\mu\nu}$ built from four plaquette loops around the point x in the $\mu-\nu$ plane (Sheikholeslami-Wohlert or *clover* term).

- c_{sw} in the SW term is a function of the bare coupling g chosen so that $O(a)$ cut-off effects cancel in on-shell quantities. $c_{\mathrm{sw}} = 1$ at tree-level in PT.

Improved Lattice Bilinears

- to improve on-shell quantities one has to improve also the local composite fields used in the correlators. Let's treat the case of the axial current

$$A_\mu^a(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a\psi(x) \qquad\qquad P^a(x) = \bar{\psi}(x)\gamma_5\tfrac{1}{2}\tau^a\psi(x)$$

  (we consider only the case of two-degenerate light quarks. In the case of non-degenerate quarks, e.g. $N_f = 2{+}1$, more improvement coefficients are required).

- the on-shell $O(a)$ improved lattice field is then given by $\phi_I(x) = \phi(x) + a\delta\phi(x)$ where $\delta\phi$ is a linear combination of a lattice representation of the linear independent fields appearing at $O(a)$.

- taking into account the transformation properties of A_μ^a under lattice symmetries and charge conjugation one finds

$$(\mathcal{O}_6)_\mu^a = \bar{\psi}(x)\gamma_5\tfrac{1}{2}\tau^a\sigma_{\mu\nu}D_\nu\psi(x) - \bar{\psi}(x)\overleftarrow{D}_\nu\sigma_{\mu\nu}\gamma_5\tfrac{1}{2}\tau^a\psi(x)$$

$$(\mathcal{O}_7)_\mu^a = \bar{\psi}(x)\gamma_5\tfrac{1}{2}\tau^a D_\mu\psi(x) + \bar{\psi}(x)\overleftarrow{D}_\mu\gamma_5\tfrac{1}{2}\tau^a\psi(x) \qquad (\mathcal{O}_8)_\mu^a = m\bar{\psi}(x)\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a\psi(x)$$

- $(\mathcal{O}_6)^a_\mu$ can be related to the other two via field equations and so may be dropped. The $O(a)$ counterterms associated to $(\mathcal{O}_8)^a_\mu$ amounts to a renormalization of A^a_μ. Since we have not imposed any renormalization condition so far we postpone the discussion of this issue. We are left with

$$\delta A^a_\mu(x) = c_A \tfrac{1}{2}(\partial^*_\mu + \partial_\mu) P^a(x)$$

- c_A depends on g and has to be chosen so as to achieve the cancellation of $O(a)$ cut-off effects. In PT it is of order g^2 because at tree-level A^a_μ is already on-shell improved (up to the mass-dependent factor mentioned above).

- For the other bilinear a similar analysis shows that

$$\delta V^a_\mu(x) = c_V \tfrac{1}{2}(\partial^*_\mu + \partial_\mu) T^a_{\mu\nu}(x) \qquad \delta P^a(x) = 0 \qquad \delta S^a(x) = 0$$
$$\delta T^a_{\mu\nu}(x) = c_T \tfrac{1}{2}\big[(\partial^*_\mu + \partial_\mu) V^a_\nu(x) - (\partial^*_\nu + \partial_\nu) V^a_\mu(x)\big]$$

where we have again neglected the mass-dependent renormalization factor.

Mass-Independent Renormalization Schemes

- In the improved theory the renormalization conditions on the gauge coupling, quark mass and improved composite fields must be chosen with care. We want correlation functions of the renormalized fields, at fixed non-zero physical distances and fixed renormalized coupling and mass, to converge to the continuum limit with rate $\propto a^2$ (after having tuned c_{sw}, c_A, c_V, c_T).

- We could impose all renormalization conditions on a set of renormalized correlation functions defined at the same point (g, am) in the bare parameter space. In this case, the $O(a)$ mass-dependent renormalization term we have dropped until now would not be necessary to have $O(a)$ improvement.

- A disavantage of this kind of schemes is however that the renormalized coupling and fields implicitly depend on the quark mass.

- Mass-independent renormalization schemes, where one imposes the renormalization conditions at zero quark mass, hare intrinsically simpler and better suited to discuss the scale evolution of the renormalized parameters. $\Rightarrow$ to obtain $O(a)$ improvement, mass-dependent renormalization factors of the bare parameters can not be ignored.

- Naive mass-independent schemes. Due to additive mass renormalization, in the plane of bare parameters a critical line $m = m_{\rm cr}(g)$ is expected to exist where the physical quark mass vanishes. We want to parametrize the theory around this line and we introduce the subtracted mass $m_{\rm q} = m - m_{\rm cr}$.

- Remark: $m_{\rm cr}(g)$ depends on how the precisely the physical quark mass is defined. Different definition lead to values of $m_{\rm cr}(g)$ which differ by $O(a)$ artifacts. In the improved theory they will be $O(a^2)$ and so we neglect them.

- Common to choice of renormalized parameters g_r and m_r in a mass-independent scheme to be related to the bare parameters through

$$g_r^2 = g^2 Z_g(g^2, a\mu) \qquad\qquad m_r = m_{\rm q} Z_m(g^2, a\mu)$$

  where μ is the renormalization scale and

$$Z_i(g^2, a\mu) = 1 + Z_i^{(1)}(a\mu) g^2 + Z_i^{(2)}(a\mu) g^4 + \dots$$

- We now show that these schemes always lead to uncancelled $O(a)$ corrections in some renormalized quantities.

- At $g=0$ (free Wilson-quark theory) we have that $m_{\rm cr}=0$ and according to the above renormalization conditions $m_r=m$. This leads however to uncancelled $O(a)$ corrections in various quantities.

- For instance in the quark *pole mass* (i.e. the energy of a free quark with zero momentum)

$$m_{\rm p}=\frac{1}{a}\ln(1+am)=m_r-\frac{1}{2}am_r^2+\ldots$$

- It is possible to correct this deficit by replacing the above renormalization condition for the mass with

$$m_r=m_{\rm q}(1-\tfrac{1}{2}am_{\rm q})+O(g^2)$$

- However this turns out to be not sufficient because the renormalization condition for g_r still give rise to other uncancelled $O(a)$ corrections. The argumentation here is more difficult because the problem shows up only at one loop in PT.

- An example where this happens is the running coupling $g_{\rm SF}(\mu)$ in the Schrödinger Functional (SF) scheme. After one-loop perturbative computation ad using the definitions above for the renormalized coupling and mass one can express $g_{\mathrm{SF}}(\mu)$ in terms of g_r. In this expression there happen to be uncancelled $O(a)$ cut-off effects.

- The two naive mass-independent renormalization conitions above have to be modified to be compatible with $O(a)$ improvement.

- Improved mass-independent schemes. We recall that complete $O(a)$ improvement require renormalization of the bare parameters by factors of the form $1 + b_i(g^2)am_{\mathrm{q}}$. We thus introduce

$$\tilde{g}^2 = g^2(1 + b_g am_{\mathrm{q}}) \qquad \tilde{m}_{\mathrm{q}} = m_{\mathrm{q}}(1 + b_m am_{\mathrm{q}})$$

  where the coefficients $b_g = b_g(g^2)$ and $b_m = b_m(g^2)$ should be chosen such to cancel any remaining $O(a)$ cut-off effect.

- The general mass-independent renormalization scheme compatible with $O(a)$ improvement is now given by

$$g_r^2 = \tilde{g}^2 Z_g(\tilde{g}^2, a\mu) \qquad m_r = \tilde{m}_{\mathrm{q}} Z_m(\tilde{g}^2, a\mu)$$

- This means that in order to reach the continumm limit without having $O(a)$ effects in the physical quantities we have to scale the bare parameters g, $m_{\rm q}$ in a $m_{\rm q}$-dependent way. Instead, $\tilde{g}$, $\tilde{m}_{\rm q}$ scale independently of $m_{\rm q}$. One can prove that $b_g = b_g(g^2)$ and $b_m = b_m(g^2)$ are independent of the particular renormalization scheme chosen.

- It is straightforward to extend the discussion to the renormalization of the local (multiplicatively renormalizable) composite fields ϕ. If ϕ_I is the improved associated field defined in a previous section, the renormalized improved field is

$$\phi_r(x) = Z_\phi(\tilde{g}^2, a\mu)(1 + b_\phi a m_{\rm q})\phi_I(x)$$

where $b_\phi = b_\phi(g^2)$ plays a role analogous to that of b_g, b_m and is independent of the renormalization condition chose to fix Z_ϕ.

- In PT $b = b^{(0)} + b^{(1)} g^2 + \dots$ where $b_g^{(0)} = 0$, $b_m^{(0)} = -\frac{1}{2}$, $b_A^{(0)} = b_P^{(0)} = 1$.

- Renormalization conditions. A complete specification of a mass-independent renorm. scheme requires the we fix the finite parts of the RCs Z_g, Z_m, Z_ϕ by imposing an appropriate set of renorm. conditions.

- Different schemes are related by transformations of the form (up to $O(a^2)$)

$$g_r^2 = g_r^2 X_g(g_r^2) \qquad m_r^2 = m_r^2 X_m(g_r^2) \qquad \phi_r^2 = \phi_r^2 X_\phi(g_r^2)$$

- In PT minimal subtraction is technically attractive. It is defined by the requirement that the expansion coefficients $Z_g^{(l)}$, $Z_m^{(l)}$, $Z_\phi^{(l)}$ are polynomials in $\ln(a\mu)$ with no constant term, to any order $l \geq 1$ of PT.

- Non-perturbatively, mass-independent renorm. schemes are not as easy to define because the correspondingly defines RCs have to be conmputable through numerical simulations. Since they refer to the properties of the teory at zero quark mass, one is then confronted with the problem of simulating QCD with very light quarks.

- RI-MOM and SF are the only two possibilities known at present. RI-MOM is simpler to implement numerically and to be matched to continuum schemes (a necessary step to compute physical observables). SF however has the advantage that is defined in finite volume, where the lattice size L is both the inverse renormalization scale and also the infrared cut-off of the theory (thus allowing to perform simulations at zero quark mass).

Twisted Mass QCD

- Twisted mass QCD (tmQCD) for two-degenerate flavors is defined by adding a *twisted mass* term to the QCD lagrangian (to be defined we will consider the Wilson discretization)

$$S_{\rm tmQCD} = a^4 \sum_x \bar\psi(x)\big[D_W + m + i\mu\gamma_5\tau^3\big]\psi(x)$$

where $D_W = \gamma_\mu(\nabla^*_\mu + \nabla_\mu) - \nabla^*_\mu\nabla_\mu$ is the Wilson-Dirac operator and ∇_μ, ∇^*_μ are the forward and backward lattice covariant derivatives.

- By performing an axial rotation of ψ and redefining m_r and μ_r

$$\psi \to \psi' = e^{i\alpha\gamma_5\frac{\tau^3}{2}}\psi \qquad \bar\psi \to \bar\psi' = \bar\psi e^{i\alpha\gamma_5\frac{\tau^3}{2}}$$

$$m'_r = m_r\cos(\alpha) + \mu_r\sin(\alpha) \qquad \mu'_r = \mu_r\cos(\alpha) - m_r\sin(\alpha)$$

one can show the following relation between renormalized correlation functions of multi-local operators $O(x_1,\dots,x_n)$ (with $O'[\bar\psi',\psi'] \equiv O[\bar\psi,\psi]$) to hold

$$\langle O_r(x_1,\dots,x_n)\rangle_{(m_r,\mu_r)} = \langle O'_r(x_1,\dots,x_n)\rangle_{(m'_r,\mu'_r)}$$

- Standard QCD ($\mu'_r = 0$) is then recovered through a rotation $\tan(\alpha) = \frac{\mu_r}{m_r}$.

- tmQCD has a series of interesting properties:
  1. the twisted mass is now an infrared cut-off and thus suitable to be simulated at small quark masses.
  2. operator mixing can be (at least in some cases) simplified.
  3. moreover, when $m_r = 0$ (i.e. $\alpha = \pi/2$) one has *automatic* $O(a)$ improvement of parity-even correlators.

- however parity and isospin are broken explicitly by the twisted mass term. They are violated only by lattice artifacts and thus recovered as $a \to 0$.

- notice that, to simulate tmQCD at a definite angle α, m_{cr} should be known together with the RCs Z_m and Z_μ. The last two are not needed only if we set $m = m_{\mathrm{cr}}$ (i.e. $\alpha = \pi/2$).

- The question of the precision to which m_{cr} has to be known than arises, expecially because in general, with the (uninproved) Wilson action, m_{cr} can be determined only up to $O(a\Lambda^2_{QCD})$ effects.

- We want to prove now the property of *automatic* $O(a)$ improvement of parity-even correlators at $\alpha = \pi/2$ and discuss the choice of m_{cr}.

Automatic $O(a)$ Improvement of tmQCD at $\alpha = \pi/2$

- Discussion based on [Frezzotti, Martinelli, Papinutto, Rossi (2005)]

- we re-write the tmQCD action choosing $m = m_{\rm cr}$ (i.e. $m_r = 0$ up to $O(a\Lambda^2_{QCD})$ effects) and rotating the basis $\psi \rightarrow \exp\{i\alpha\gamma_5\frac{\tau^3}{2}\}\psi$ with $\alpha = \pi/2$

$$S_{\rm tmQCD} = a^4 \sum_x \bar{\psi}(x)\Big[\gamma_\mu(\nabla_\mu + \nabla^*_\mu) + \mu - i\gamma_5\tau^3\Big(- a\tfrac{1}{2}\nabla^*_\mu\nabla_\mu + m_{\rm cr}\Big)\Big]\psi(x)$$

- this basis makes it evident that at $\alpha = \pi/2$ the physical quark mass is proportional to the twisted mass μ while the standard Wilson mass is set to the value needed to tune $m_r = 0$.

- The lattice artifacts are given by a rotated (now parity-odd) Wilson term and thus are different (as also the renormalization properties) with repsect to standard Wilson theory. This is the reason why tmQCD become interesting.

- The Symanzik local effective theory (LET) of tmQCD at $\alpha = \pi/2$ is

$$S_{\rm tmQCD} = \int d^4y \left[\mathcal{L}_0(y) + \sum_{k=0}^{\infty} a^{2k+1}\mathcal{L}_{2k+1}(y) + \sum_{k=1}^{\infty} a^{2k}\mathcal{L}_{2k}(y)\right]$$

where, as before, $\mathcal{L}_0$ is the continuum QCD lagrangian

- The spurionic symmetry $\mathcal{D}_d \times P \times (\mu \to -\mu)$ (P is the parity operator and $\mathcal{D}_d$ acts on an operator of dimension d multiplying it by the phase $\exp(i\pi d) = (-1)^d$ and reflecting its space-time arguments) $\Rightarrow \mathcal{L}_{2k}$ are parity even and iso-singlets, $\mathcal{L}_{2k+1}$ are parity odd and twisted in iso-spin space. We define

$$\mathcal{L}_{\rm odd} \equiv \sum_{k=0}^{\infty} a^{2k+1}\mathcal{L}_{2k+1}\,, \qquad \mathcal{L}_{\rm even} \equiv \sum_{k=1}^{\infty} a^{2k}\mathcal{L}_{2k}\,.$$

- $\mathcal{L}_1$, the term of $O(a)$, can be shown to be a linear combination $\mathcal{L}_1 = \delta_{1,sw}\mathcal{L}_{1,sw} + \delta_{1,\mu}\mathcal{L}_{1,\mu} + \delta_{1,e}\mathcal{L}_{1,e}$ where

$$\mathcal{L}_{1,sw} = \tfrac{i}{4}\bar\psi\gamma_5\tau_3\sigma_{\mu\nu}F_{\mu\nu}\psi\,, \quad \mathcal{L}_{1,\mu} = \mu^2\bar\psi i\gamma_5\tau_3\psi\,, \quad \mathcal{L}_{1,e} = \Lambda^2_{QCD}\bar\psi i\gamma_5\tau_3\psi\,,$$

- $\mathcal{L}_{1,e}$ is needed to describe the $O(a)$ uncertainties in the estimation of $m_{\rm cr}$.

- Both $\mathcal{L}_{1,sw}$ and $\mathcal{L}_{1,e}$ are absent if the SW term is introduced with c_{sw} determined non-perturbatively and $m_{\rm cr}$ set to its $O(a)$ improved value.

- We consider the renormalized lattice correlation function $G_O(x)$ of the product of local, multiplicativly renormalizable operators $O(x) \equiv \prod_{j=1}^{n} O_j(x_j)$ (where as usual we keep the points $x_1, \ldots, x_n$ at non zero physical distances from each other) which globally has vacuum quantum numbers.

- Let's consider the Symanzik continuum LET of $G_O(x)$

$$G_O(x) = \langle [O(x) + \Delta_{\rm odd} O(x) + \Delta_{\rm even} O(x)] e^{-\int d^4y [\mathcal{L}_{\rm odd}(y) + \mathcal{L}_{\rm even}(y)]} \rangle$$

where $\mathcal{L}_{\rm odd} = \mathrm{O}(a)$ and $\mathcal{L}_{\rm even} = \mathrm{O}(a^2)$ and

$$\Delta_{\rm odd} O = \sum_{k=0}^{\infty} a^{2k+1} \delta_{2k+1} O \,, \qquad \Delta_{\rm even} O = \sum_{k=1}^{\infty} a^{2k} \delta_{2k} O \,.$$

appear in the Symanzik expansion of the operator O and they are eventually redefined in order to to regularize terms where a parity-odd (resp. parity-even) product of $\mathcal{L}_{\rm odd}$ and/or $\mathcal{L}_{\rm even}$ insertions come in contact with some of the points where the local factors of O are concentrated.

- the $\delta_p O$'s can be viewed as the set of n-point operators of dimension $\dim O + p$ necessary to improve O at order a^p. It is easily proved that $\delta_{2k} O$ has the same parity of O while $\delta_{2k+1} O$ has opposite parity.

- *Automatic* $O(a)$ improvement of the correlator of a globally parity-even operator $O(x)$ is easily proved from the parity properties of $\mathcal{L}_{\rm odd}$, $\mathcal{L}_{\rm even}$, $\Delta_{\rm odd}O$ and $\Delta_{\rm even}O$

$$G_O(x) = \langle O(x)\rangle + a^2\langle O_2(x)\rangle + a^4\langle O_4(x)\rangle + \mathrm{O}(a^6)$$

where, for instance

$$\langle O_2(x)\rangle = \langle \delta_2 O(x)\rangle + \int d^4y\,\langle \delta_1 O(x)\mathcal{L}_1(y)\rangle + \\ + \int d^4y\,\langle O(x)\mathcal{L}_2(y)\rangle + \tfrac{1}{2}\int d^4y \int d^4z\,\langle O(x)\mathcal{L}_1(y)\mathcal{L}_1(z)\rangle$$

- In fact, we have proved not only the $O(a)$ improvement of the theory but also the fact the all the $O(a^{2k+1})$ lattice artifacts are absent.

- Full analysis of cutoff effects beyond $O(a)$ is extremely complicated. We are interested here only in terms that are enhanced by $1/m_\pi^2$ poles. In particular, at a fixed order a^{2k}, in the contributions with the highest multiplicity of pion poles, which correspond to the leading *chirally enhanced* cutoff effects.

- The following result can be proved: at order a^{2k} the term with the highest multiplicity of pion poles give contributions with a $2k$-fold pion pole of the form

$$\left[\left(\frac{1}{m_\pi^2}\right)^{2k}\left|\langle\Omega|\mathcal{L}_{\rm odd}|\pi^0(\mathbf{0})\rangle\right|^{2k}\mathcal{M}[O;(\pi^0(\mathbf{0}))^{2k}]\right]$$

where $\mathcal{M}[O;(\pi^0(\mathbf{0}))^{2k}]$ denotes the (sum of the) $2k$-particle matrix elements of the operator O, with each external particle being a neutral pion at zero three-momentum.

- The proof relies on the LSZ reduction formula and the observation that $\mathcal{L}_{\rm odd}$ has the continuum quantum numbers of the neutral pion.

- besides the leading "IR divergent" lattice artifacts, there are less important contributions of the type $a^{2k}/(m_\pi^2)^h$, with $2k > h \geq 1$.

- The presence of pion poles in the Symanzik expansion does not mean that the latter diverges as $\mu \to 0$. These poles will in fact rearrange in order to give the trigonometric factors in the observables related to an *imprecise* twisted angle $\alpha = \arctan(\frac{\mu_r}{m_r})$ (where $m_r = O(a\Lambda^2_{QCD})$ and $\mu_r \propto m_\pi^2$) which is not exactly equal to $\pi/2$.

- It shows instead that more μ (and thus m_π^2) becomes small more the tuning of the twist angle to $\pi/2$ worsen and large $O(a^2/\mu)$ lattice artifacts appear.

- So, as long as the inequality $m_\pi^2 \sim \mu > a\Lambda_{QCD}^2$ is not satisfied, these large artifacts will possibly lead to a breakdown of the expansion.

- Leading *chirally enhanced* lattice artifacts are proportional to powers of $\langle\Omega|\mathcal{L}_{\rm odd}|\pi^0(\mathbf{0})\rangle$ that is, at leading order in a, to powers of $\langle\Omega|\mathcal{L}_5|\pi^0(\mathbf{0})\rangle \Rightarrow$ eliminated if we can set $\langle\Omega|\mathcal{L}_5|\pi^0(\mathbf{0})\rangle = 0$.

- First way: introduce the SW term with c_{SW} non-perturbatively determined and use an $O(a)$ improved estimate of $m_{\rm cr} \Rightarrow \mathcal{L}_{1,sw}$ and $\mathcal{L}_{1,e}$ are absent and $\mathcal{L}_1$ simply proportional to $\mathcal{L}_{1,\mu} \Rightarrow$ *chirally enhanced* terms become regular $(a\mu^2/m_\pi^2)^{2k} \simeq (a\mu)^{2k}$.

- Comments: adding the SW term does not affect the results about *automatic* $O(a)$ improvement; the value of c_{sw} is precisely that for the untwisted Wilson action.

- Second way: fix $m_{\rm cr}$ through

$$\lim_{\mu\to 0}\langle\Omega|\mathcal{L}_{\rm odd}(0)|\pi(\mathbf{0})\rangle = 0 \qquad\qquad \Rightarrow$$
$$\delta_{1,e}\Lambda^2_{QCD}\langle\Omega|\bar\psi i\gamma_5\tau_3\psi|\pi\rangle = \delta_{1,sw}\tfrac{i}{4}\langle\Omega|\bar\psi\gamma_5\tau_3\sigma_{\mu\nu}F_{\mu\nu}\psi|\pi\rangle \quad \text{as} \quad \mu\to 0$$

- It turns out that this tuning of $m_{\rm cr}$ can be actually implemented by demanding restoration of parity in the chiral limit (porposed also by Sharpe using ChPT)

$$\lim_{\mu\to 0} a^3\sum_{\mathbf{x}}\langle V_0^2(x)P^1(0)\rangle = 0$$

- left-over *chirally enhanced* lattice artifacts turn out to be of the form $a^{2k}/(m_\pi^2)^{k-1}$, with both the methods proposed $\Rightarrow$ to be satisfied in order to avoid large cutoff effects becomes now

$$m_\pi^2 \sim \mu > a^2\Lambda^3_{QCD}$$

- It is possile to show that the *chirally enhanced* cut-off effects are reasonably small even if one determines $m_{\rm cr}(\mu)$ for $\mu = \mu_{\rm min}$ (the smallest mass at which the simulation is performed) without actually taking the limit $\mu \to 0$.

Numerical Determination of $m_{\rm cr}$

- Determination of $\frac{1}{\kappa_{\rm cr}(\mu)} \propto am_{\rm cr}(\mu)$ and its extrapolation $\mu \to 0$. Results in the quenched approximation from [Jansen, Papinutto, Shindler, Urbach, Wetzorke (2005)]

- Comparison of numerical results for f_π with two choices of $m_{\rm cr}$.