Lattice QCD and Non-perturbative Renormalization

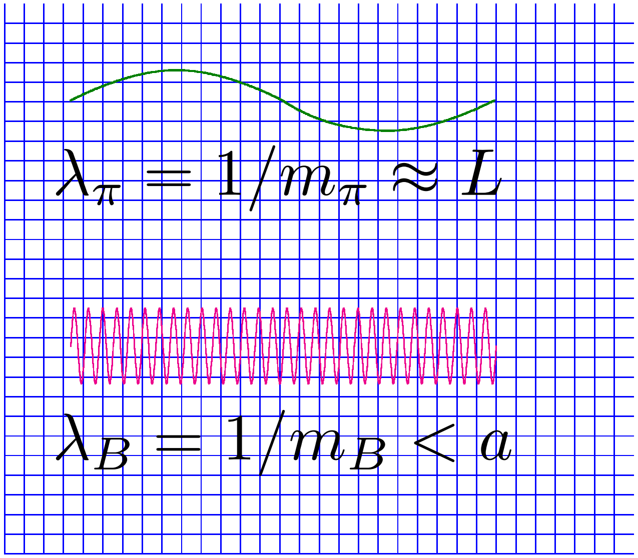
Mauro Papinutto

LPSC Grenoble
Laboratoire de Physique
Subatomique et de Cosmologie

GDR “Calculs sur réseau”, SPhN Saclay, March 4th 2009

Lecture 5: Non-Perturbative Heavy Quark Effective Theory

HQET: motivations



light quarks are too light $\Rightarrow$ extrapolate by matching with chiral effective theory.

b-quark is too heavy ($m_\mathrm{b}a > 1$) $\Rightarrow$ need an effective theory for the b quark: HQET (expansion in inverse powers of the mass of the heavy quark).

Three possible strategies:

1. Work in the static limit of HQET ($m_\mathrm{b} \to \infty$) and compute $1/m_\mathrm{b}^n$ corrections.
2. Combine relativistic simulations with $m_\mathrm{q} \approx m_\mathrm{c}$ and the static limit of HQET to interpolate at m_b.
3. Use finite size methods to relate relativistic observables computed on small volume at physical m_b to their value in infinite volume and eventually combine with HQET.

Non-perturbative HQET

heavy quark and anti-quark fields $\psi_{\rm h}$, $\psi_{\bar{\rm h}}$ are now independent and satisfy

$$P_+\psi_{\rm h} = \psi_{\rm h}\,, \quad \overline{\psi}_{\rm h}P_+ = \overline{\psi}_{\rm h}\,, \quad P_+ = \tfrac{1}{2}(1+\gamma_0)$$
$$P_-\psi_{\rm h} = \psi_{\rm h}\,, \quad \overline{\psi}_{\rm h}P_- = \overline{\psi}_{\rm h}\,, \quad P_- = \tfrac{1}{2}(1-\gamma_0)$$

HQET action on a lattice (we consider only the heavy quark fields for simplicity)

$$S_{\rm HQET} = a^4 \quad \sum_x \quad \{\overline{\psi}_{\rm h}(x)[D_0 + m_{\rm bare}]\psi_{\rm h}(x)$$
$$+\omega_{\rm spin}\underbrace{\overline{\psi}_{\rm h}(-\sigma\cdot\mathbf{B})\psi_{\rm h}}_{\mathcal{O}_{\rm spin}} + \omega_{\rm kin}\underbrace{\overline{\psi}_{\rm h}(-\tfrac{1}{2}\mathbf{D}^2)\psi_{\rm h}}_{\mathcal{O}_{\rm kin}} + {\rm O}(1/m^2)\}$$

also the composite fields have a $1/m$ expansion in the effective theory

$$A_0^{\rm HQET}(x) = Z_{\rm A}^{\rm HQET}\overline{\psi}_{\rm l}(x)\gamma_0\gamma_5\psi_{\rm h}(x) + c_{\rm A}^{\rm HQET}\overline{\psi}_{\rm l}\gamma_j\overleftarrow{D}_j\psi_{\rm h} + {\rm O}(1/m^2)$$

where $\quad \omega_{\rm kin} = {\rm O}(1/m)\,, \qquad \omega_{\rm spin} = {\rm O}(1/m)\,, \qquad c_{\rm A}^{\rm HQET} = {\rm O}(1/m)$

in the path integral: expand the action and the operators in $1/m$

$$S_{\rm HQET} = a^4 \sum_x \left\{ \mathcal{L}_{\rm stat}(x) + \sum_{\nu=1}^{\infty} \mathcal{L}^{(\nu)}(x) \right\} \qquad \mathcal{O}(x) = \mathcal{O}_{\rm stat}(x) + \sum_{\nu=1}^{\infty} \delta\mathcal{O}^{(\nu)}(x)$$

where $\mathcal{L}^{(\nu)}(x) = {\rm O}(1/m^\nu)$ and $\delta\mathcal{O}^{(\nu)}(x) = {\rm O}(1/m^\nu)$ have to be considered in the path integral only as operator insertions

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int {\rm D}\phi \, {\rm e}^{-S_{\rm rel} - S_{\rm stat}} \left\{ \mathcal{O}_{\rm stat} + \delta\mathcal{O}^{(1)} + \ldots \right\} \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \ldots \right\}$$

With this definition of the effective theory we have (at a given order in $1/m$)

- renormalizability $\equiv$ existence of the continuum limit

- continuum asymptotic expansion in $1/m$

Note that these properties are not automatic for an effective field theory. ChPT shares these properties; NRQCD does not. In the latter, the $O(1/m)$ term is kept in the leading order action used to compute the path-integral, leading to a truly *non-renormalizable* theory.

Difference between HQET and ChPT: as $1/m \to 0$ interactions are not turned off $\Rightarrow$ results in ChPT can be worked out analitically in PT while in HQET we still need the lattice formulation to evaluate non-perturbatively the lowest order theory (called the static approximation).

The terms in the effective theory are organized just by their mass dimension. The expectation that the effective field theory has a continuum limit (i.e. is non-perturbatively renormalizable) is thus the usual expectation that composite operators mix only with operators of the same or lower dimension.

Same argumentation as for Symanzik's discussion of cut-off effects of lattice filed theories $\Rightarrow$ in general the $1/m$ and the a-expansion are not independent but have to be considered as a single expansion in term of the dimension of the local fields.

If we start with a set of operators indentified by the formal continuum $1/m$ expansion, these operators will mix under renormalization with all the operators of the same or lower dimension which are allowd by the lattice symmetries and not only those allowed by the continuum symmetries.

One thus has to start from the beginning with the complete basis of operators allowed by the lattice symmetries. In other word the power counting is $a =$

$O(1/m)$. In particular, in order to go to $O(1/m)$ $S_{\rm rel}$ has to be $O(a)$ improved.

The terms that have to be taken into account are restricted by HQET lattice symmetries: 3-dim cubic group and, at lowest order in $1/m$, heavy quark spin symmetry and the local conservation of the heavy quark number (this simplify $O(a)$ improvement).

As in the case of Symanzik improvement, we are interested only in on-shell quantities and we can therefore keep all the fields at non-zero physical distance $\Rightarrow$ we can use the equation of motions derived from $S_{\rm rel} + S_{\rm stat}$ to reduce the set of operators to be considered.

Renormalisation of HQET has to be done non-perturbatively: hard cut-off $(1/a)$

$$\Rightarrow \text{e.g.} \qquad {\mathcal{O}_{\rm R}}^{\rm d=5} = Z_{\mathcal{O}} \left[\mathcal{O}^{\rm d=5} + \sum_k c_k {\mathcal{O}_k}^{\rm d=4} \right] \qquad\qquad c_k = \frac{c_k^{(0)} + c_k^{(1)} g^2 + \ldots}{a}$$

if c_k computed at a finite order l in g^2, there is no continuum limit!

$$\Delta c_k \sim \frac{g^{2(l+1)}}{a} \sim \frac{1}{a\,[\ln(a\Lambda)]^{l+1}} \;\xrightarrow{a\to 0}\; \infty$$

perturbative remainder of some parameters computed to l-loops with l arbitrary

Matching between QCD and HQET

We expect, at a given order n in HQET, the following relation between QCD and HQET observables

$$\Phi^{\mathrm{QCD}}(M) = \Phi^{\mathrm{HQET}}(M) + O\left(\tfrac{1}{M^{n+1}}\right)$$

where M is the RGI heavy quark mass.

we want to fix the bare couplings of HQET such that this equivalence between HQET and QCD is true at a order n in the $1/m$-expansion. First of all we have to fix all the parameters of QCD by requiring a set of observables (e.g. hadron masses) to agree with experiment.

Next, we determine the bare couplings of HQET at order n $(m_{\mathrm{bare}}, \omega_{\mathrm{kin}}, \omega_{\mathrm{spin}}, c_{\mathrm{A}}^{\mathrm{HQET}}, Z_{\mathrm{A}}^{\mathrm{HQET}}, \ldots)$ by imposing

$$\Phi_k^{\mathrm{QCD}}(M) = \Phi_k^{\mathrm{HQET}}(M)\,, \quad k = 1, 2, \ldots, N_n^{\mathrm{HQET}}$$

The HQET results are then correct up to $O(1/m^{n+1}) = O(M^{-(n+1)} a M^k)$ with $k = 0, \ldots, n+1$. For instance, working at order $1/m$ will give $1/M^0$ terms with $O(a^2)$ errors and $1/M$ terms with $O(a)$ errors.

In principle each $\Phi_k^{\rm HQET}(M)$ could be determined from a physical, experimentally accessible observable. However this would reduce the predictive power of the theory, since it contains more parameters of QCD (increasing the order n of the $1/m$-expansion we then would need to use more and more experimental observables.

To preserve the predictability of the theory, we may instead use quantities $\Phi_k^{\rm QCD}(M)$ comuted in the continuum limit of lattice QCD. This require to treat the heavy quark as a relativistic particle on the lattice, which was exactly the problem in the motivations for using HQET! we will explain in a moment how this apparent contraddiction can be solved.

For a $32^3 \times 64$ lattice with a physical volume of $2\,\rm fm$ (above which finite volume effects are considered to be negligible) the corresponding lattice spasing is $a \approx 0.06\,\rm fm$ and the subtracted bare $\rm b$-quark mass is $am_{\rm q} \approx 1 \Rightarrow$ QCD lattice artifacts are expected to be very large.

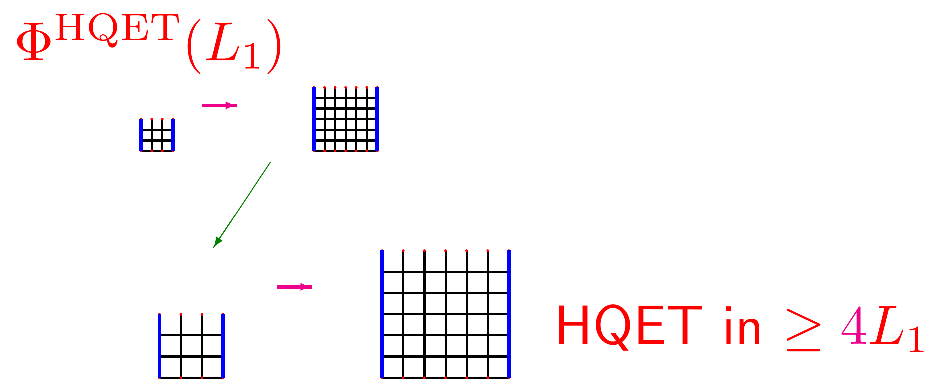
Way out: consider observables Φ_k defined in finite (small) volume and use the fact that the parameters of the QCD and HQET lagrangians are independent of the volume.

Non-perturbative matching between QCD and HQET

The Φ_k are defined on $L = L_1 \ll 2\,\mathrm{fm}$ where $L_1 \approx 0.4\,\mathrm{fm}$. We can thus simulate very fine a's where $m_\mathrm{b} a \ll 1$ and $1/(m_b L_1) \ll 1$ (the last condition is needed to have a well behaved $1/m$-expansion in finite volume and small size of the residual $O(1/m^{n+1})$ corrections).

HQET-parameters from QCD observables in small volume at small lattice spacing (using the Schrödinger Functional)

Physical observables (e.g. $B_{\mathrm{B_s}}$, $F_{\mathrm{B_s}}$) need a large volume, such that the B-meson fits comfortably: $L \approx 4L_1 \approx 1.6\,\mathrm{fm}$



Connection achieved recursively with the step-scaling function method:

$$\Phi_k^{\mathrm{HQET}}(2L) = F_k\left(\left\{\Phi_j^{\mathrm{HQET}}(L), j = 1, \ldots, N_n^{\mathrm{HQET}}\right\}\right)$$

fully non-perturbative formulation of HQET (including matching) [Heitger & Sommer, 2004]

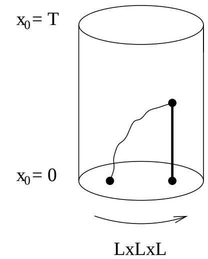
continuum limit can be taken in all steps

Example: $M_{\rm b}$ static (at order $1/m^0$)

[Heitger & Sommer, 2004; M. Della Morte, N. Garron, M. Papinutto and R. Sommer, 2005-2006]

finite volume B-meson "mass":

$$\Gamma = -\partial_0 \log[f_{\rm A}(x_0)]_{x_0=L/2,T=L}$$

$f_{\rm A} \sim$ 

matching condition in finite volume:

$$L_1\Gamma(L_1,M_{\rm b}) \equiv \Phi_2^{\rm QCD}(L_1,M_{\rm b}) = \Phi^{\rm HQET}(L_1,m_{\rm bare}) \equiv L_1\left[\Gamma^{\rm stat}(L_1)+m_{\rm bare}\right]$$

∞ volume equation ($m_{\rm bare}$ eliminated and $E_{\rm stat} = \lim_{L\to\infty}\Gamma^{\rm stat}(L)$):

$$L_1 m_{\rm B} = L_1 E_{\rm stat} + L_1 m_{\rm bare}$$

$$= L_1 E_{\rm stat} - L_1\Gamma^{\rm stat}(L_1) + \Phi_2^{\rm QCD}(L_1,M_{\rm b})$$

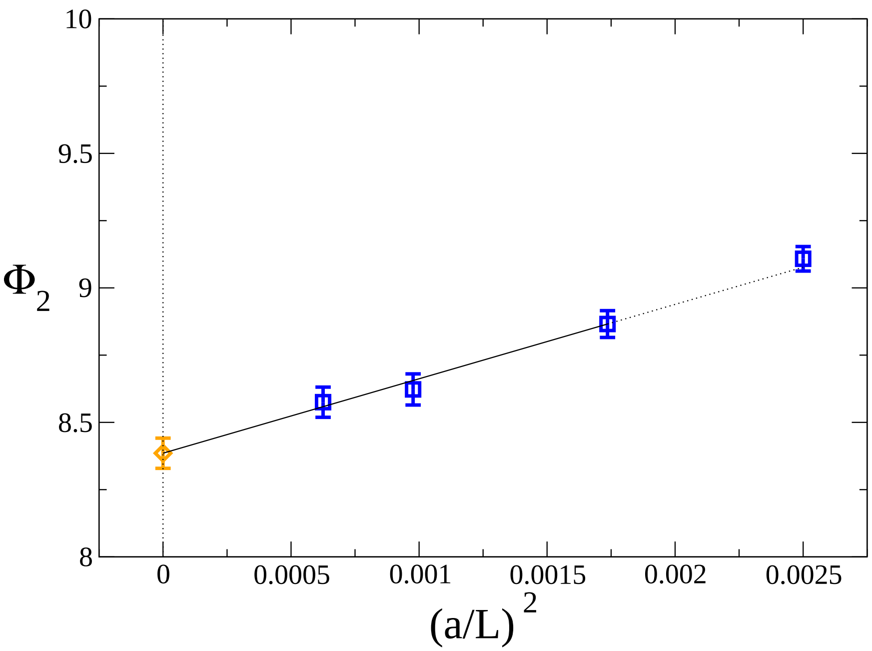
$$= L_1 E_{\rm stat} - L_1\Gamma^{\rm stat}(2L_1) + \underbrace{L_1\Gamma^{\rm stat}(2L_1) - L_1\Gamma^{\rm stat}(L_1)}_{=\frac{1}{2}\sigma_{\rm m}(\bar{g}^2(L_1))} + \Phi_2^{\rm QCD}(L_1,M_{\rm b})$$

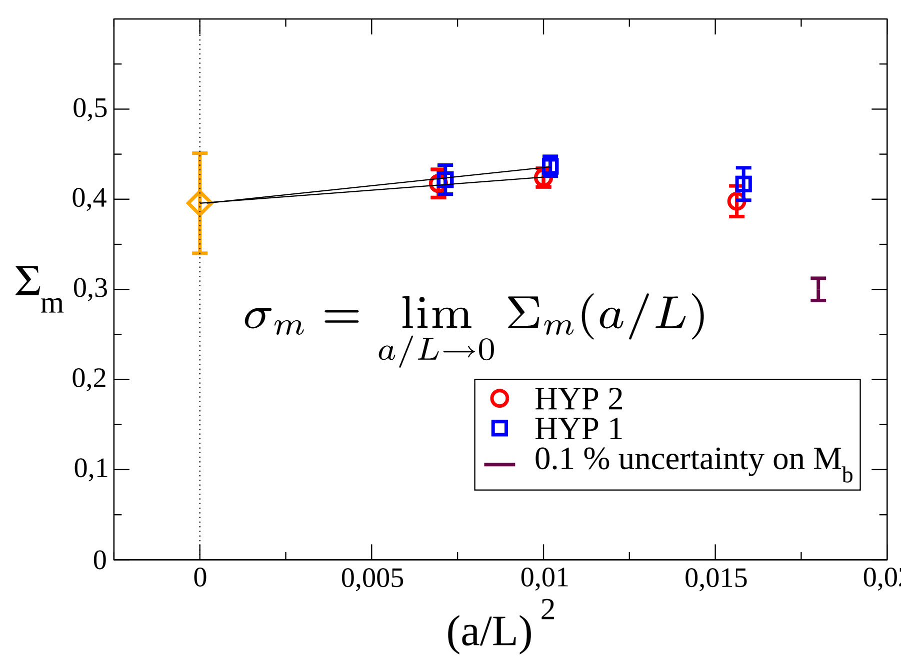
$$\underbrace{L_1 m_{\rm B}}_{\rm exp.} = \underbrace{L_1[E_{\rm stat} - \Gamma^{\rm stat}(2L_1)]}_{a\to 0 \text{ in HQET}} + \underbrace{\tfrac{1}{2}\sigma_{\rm m}(u_1)}_{a\to 0 \text{ in HQET}} + \underbrace{\Phi_2^{\rm QCD}(L_1,M_{\rm b})}_{a\to 0 \text{ for } M_{\rm b}L_1 \gg 1}$$

In $E_{\rm stat}$ and $\Gamma^{\rm stat}(2L_1)$ (and analogously in $\Gamma^{\rm stat}(2L_1)$ and $\Gamma^{\rm stat}(L_1)$) there are $1/a$ power divergences that cancel in the difference if the same a is chosen.

experiment	Lattice with $am_{\mathrm{q}} \ll 1$

$$
\begin{array}{ccc}
m_{\mathrm{B}} = 5.4\,\mathrm{GeV} & & \Gamma(L_1, M) \\
\downarrow & \begin{array}{c} L_2 = 2L_1 \\ u_i = \bar{g}^2(L_i) \end{array} & \downarrow \\
\Gamma^{\mathrm{stat}}(2L_1) & \xleftarrow[\sigma_{\mathrm{m}}(u_1)]{} & \Gamma^{\mathrm{stat}}(L_1)
\end{array}
$$





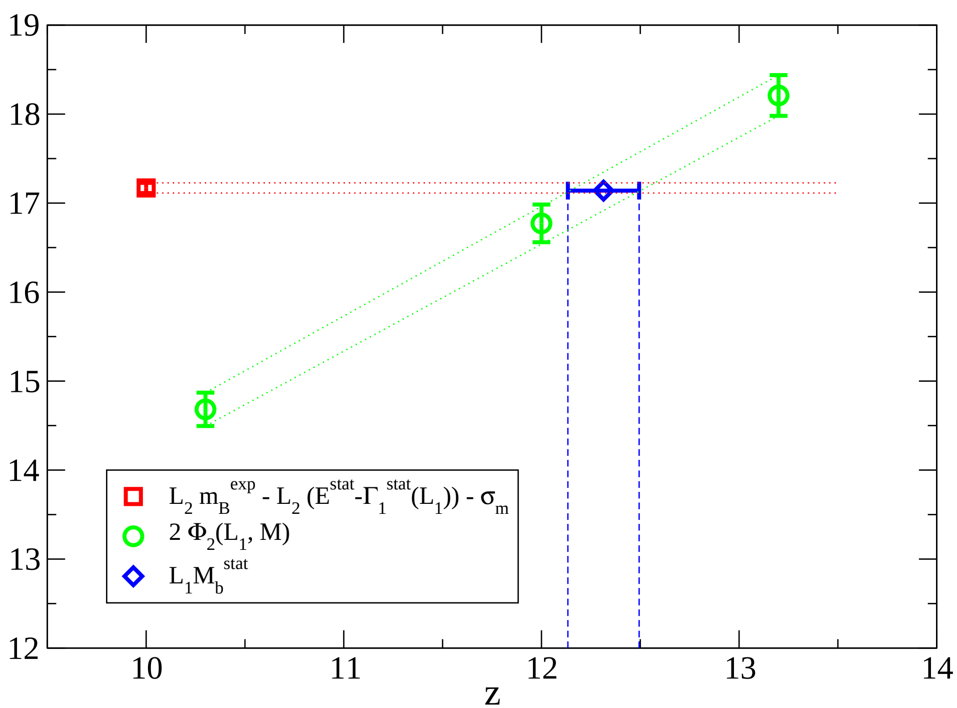
Set up: light quark mass set to zero on L_1; 3 values of the RGI heavy quark mass around M_{b}; light quark mass set to the strange value on $4L_1$.

$M_{\rm b}$ static

$$\underbrace{L_1 m_{\rm B}}_{\text{experiment}} - L_1[E_{\rm stat} - \Gamma^{\rm stat}(2L_1)] - \tfrac{1}{2}\sigma_{\rm m}(\bar{g}^2(L_1)) = \Phi_2^{\rm QCD}(L_1, M_{\rm b})$$

Solve the above equation for $M_{\rm b}$ (the RGI b-quark mass).

At this order, $M_{\rm b}$ is affected by $O(\frac{1}{L_1^2 M_{\rm b}})$, $O(\frac{\Lambda_{QCD}}{L_1 M_{\rm b}})$,$O(\frac{\Lambda_{QCD}^2}{M_{\rm b}})$ errors. For our choice $L_1 = 0.4\,{\rm fm}$ they turn out to be of the same order of magnitude.



$M_b^{\rm stat} = 6771 \pm 99$ MeV

and obtain the slope

$$S = \frac{1}{L_1}\frac{\partial \Phi_2^{\rm QCD}(L_1, M)}{\partial M} = 0.61(5)$$

error dominated by that on renorm. constant of the quark mass.

$M_{\rm b}$ at order $1/m$ [M. Della Morte, N. Garron, M. Papinutto and R. Sommer, 2005-2006]

coefficients in the action:

$$\begin{array}{lll} \mathrm{O}(1) & m_{\rm bare} & \\ \mathrm{O}(1/m) & \omega_{\rm kin} & \text{of } \overline{\psi}_{\rm h}(-\frac{1}{2}\mathbf{D}^2)\psi_{\rm h} \\ \mathrm{O}(1/m) & \omega_{\rm spin} & \text{of } \overline{\psi}_{\rm h}(-\sigma\cdot\mathbf{B})\psi_{\rm h} \end{array}$$

$\omega_{\rm spin}$ cancels in spin averaged quantities.

∞ volume $\quad m_{\rm B} = E^{\rm stat} + m_{\rm bare} + \omega_{\rm kin} E^{\rm kin}$

Matching 1 $\quad \Gamma^{\rm QCD}(L, M_{\rm b}) = \Gamma^{\rm stat}(L) + m_{\rm bare} + \omega_{\rm kin}\Gamma^{\rm kin}(L) = \frac{\Phi_2^{\rm HQET}}{L}$

Matching 2 $\quad \Phi_1^{\rm QCD}(L, M_{\rm b}) = \omega_{\rm kin} R_1^{\rm kin}(L) = \Phi_1^{\rm HQET}$

$$m_{\rm B} = \left[E^{\rm stat} - \Gamma^{\rm stat}(L)\right] + \Gamma^{\rm QCD}(L, M_{\rm b}) + \left[\frac{\Phi_1^{\rm QCD}(L,M_{\rm b})}{R_1^{\rm kin}(L)}(E^{\rm kin} - \Gamma^{\rm kin}(L))\right]$$

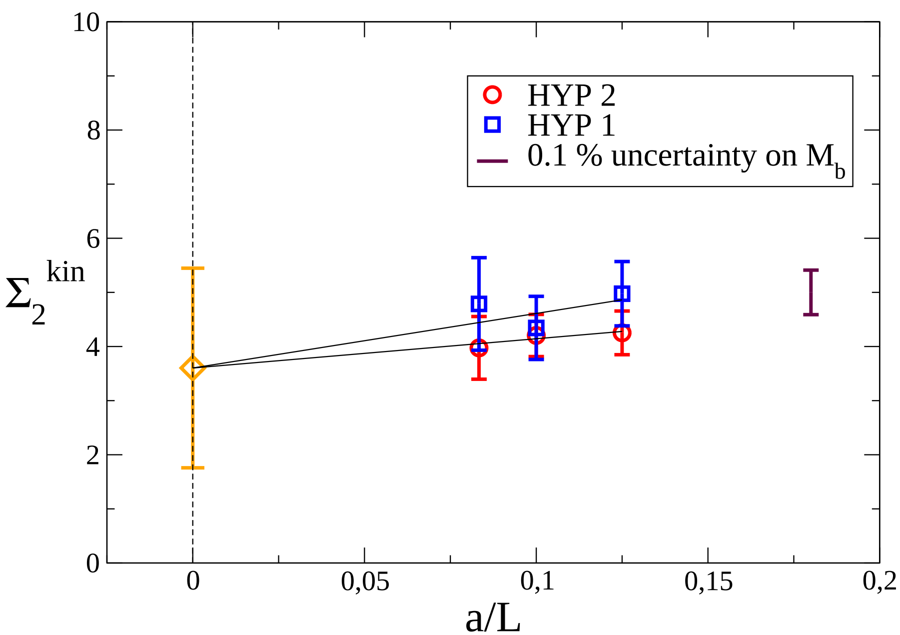
($m_{\rm bare}$, $\omega_{\rm kin}$ eliminated). Set $L = L_1$ and insert $0 = \Gamma^{\rm stat}(2L_1) - \Gamma^{\rm stat}(2L_1)$ and $0 = \Gamma^{\rm kin}(2L_1) - \Gamma^{\rm kin}(2L_1)$. The continuum limit can be taken separately in every part of the formula connecting L and $2L$ (in which the power divergences cancel because the same lattice spacing can be chosen). One obtain the SSF:

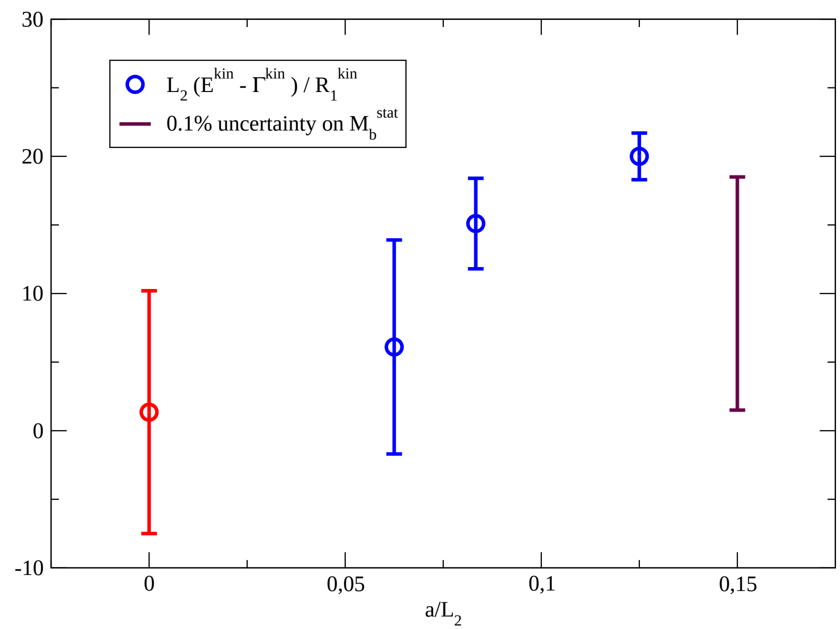
$\Phi_1(2L) = \sigma_1^{\rm kin}(u)\Phi_1(L)$, $\qquad \Phi_2(2L) = 2\Phi_2(L) + \sigma_{\rm m}(u) + \sigma_2^{\rm kin}(u)\Phi_1(L)$

$$\Rightarrow \qquad m_{\mathrm{B}} = m_{\mathrm{B}}^{\mathrm{stat}} + m_{\mathrm{B}}^{(1a)} + m_{\mathrm{B}}^{(1b)}\,, \qquad m_{\mathrm{B}}^{\mathrm{exp}} = m_{\mathrm{B}}(M_{\mathrm{b}}^{\mathrm{stat}} + M_{\mathrm{b}}^{(1a)} + M_{\mathrm{b}}^{(1b)})$$

Most difficult steps of the computation:

$$m_{\mathrm{B}}^{(1a)}(M) = \frac{1}{2L_1}\sigma_2^{\mathrm{kin}}(u_1)\,\Phi_1(L_1, M) \qquad m_{\mathrm{B}}^{(1b)}(M) = \frac{(E^{\mathrm{kin}} - \Gamma_1^{\mathrm{kin}}(2L_1))}{R_1^{\mathrm{kin}}}\Phi_1(2L_1, M)$$





In $\sigma_2^{\mathrm{kin}}(u_1)$ and $(E^{\mathrm{kin}} - \Gamma_1^{\mathrm{kin}}(2L_1))$ cancellation of $1/a^2$ power divergences (extrapolation linear in a)

Results at order $1/m$

then the $1/m$ correction to $M_{\mathrm{b}}^{\mathrm{stat}}$ $\quad (M_{\mathrm{b}}^{(1)} = M_{\mathrm{b}}^{(1a)} + M_{\mathrm{b}}^{(1b)})$ is

$$M_b^{(1\mathrm{a})} \;=\; -\frac{\sigma_2^{\mathrm{kin}}(\bar{g}^2(L_1))\Phi_1(L_1, M_b^{\mathrm{stat}})}{S\,2L_1} = -30(15)\,\mathrm{MeV}$$

$$M_b^{(1\mathrm{b})} \;=\; -\frac{(E^{\mathrm{kin}}-\Gamma_1^{\mathrm{kin}}(2L_1))\Phi_1(2L_1,M)}{SR_1^{\mathrm{kin}}} = -5(33)\,\mathrm{MeV}$$

and in the $\overline{\mathrm{MS}}$ scheme $m_{\mathrm{b}}(m_{\mathrm{b}}) = m_{\mathrm{b}}^{\mathrm{stat}} + m_{\mathrm{b}}^{(1)}$

$$m_{\mathrm{b}}^{\mathrm{stat}} \;=\; 4.35(6)\,\mathrm{GeV}\,, \quad m_{\mathrm{b}}^{(1)} = -0.02(2)\,\mathrm{GeV}\,.$$

which agrees with PDG, despite quenched approximation.

At this order, M_{b} is affected by $O(\frac{1}{L_1^3 M_{\mathrm{b}}^2})$, $O(\frac{\Lambda_{QCD}}{L_1^2 M_{\mathrm{b}}^2})$, $O(\frac{\Lambda_{QCD}^2}{L_1 M_{\mathrm{b}}^2})$, $O(\frac{\Lambda_{QCD}^3}{M_{\mathrm{b}}^2})$ errors. For $L_1 = 0.4\,\mathrm{fm}$ they are of the same order of magnitude and very small.

This has been check by using different matching conditions: f_{A} needs $\mathrm{O}(1/m)$-correction to A_0^{stat} $\Rightarrow$ more step scaling functions $\Rightarrow$ final result agrees up to (small) $\mathrm{O}(1/m^2)$-corrections which turn out to be exactly of $O(\Lambda_{QCD}^3/M_{\mathrm{b}}^2)$

Non-perturbative renormalisation of HQET operators

[Palombi,Papinutto,Pena,Wittig, 2005-2007 + Dimopoulos,Herdoiza,Palombi,Papinutto,Pena,Vladikas,Wittig 2007]

$\Delta B = 2$ oscillations: $\langle \bar{B}_q^0 | \mathcal{O}_{\rm LL}^{\Delta B=2} | B_q^0 \rangle = \frac{8}{3} B_{B_q} f_{B_q}^2 m_{B_q}^2$ relevant for UT analysis

In HQET at leading order (static approximation) one has

$$\begin{aligned}\langle \bar{B}_q^0 | \mathcal{O}_{\rm LL}^{\Delta B=2}(m_{\rm b}) | B_q^0 \rangle &= C_1(m_{\rm b},\mu)\, \langle \bar{B}_q^0 | \hat{Q}_1^+(\mu) | B_q^0 \rangle_{\rm HQET} \\ &+ C_2(m_{\rm b},\mu)\, \langle \bar{B}_q^0 | \hat{Q}_2^+(\mu) | B_q^0 \rangle_{\rm HQET} + {\rm O}\left(1/m_{\rm b}\right)\end{aligned}$$

HQET four-fermion operator basis:

$$\mathcal{O}^{\pm}_{\Gamma_1\Gamma_2} = \frac{1}{2}\left[(\bar{\psi}_{\rm h}\Gamma_1\psi_1)(\bar{\psi}_{\bar{\rm h}}\Gamma_2\psi_2) \pm (\bar{\psi}_{\rm h}\Gamma_1\psi_2)(\bar{\psi}_{\bar{\rm h}}\Gamma_2\psi_1)\right]$$

$$(Q_1^+, Q_2^+, Q_3^+, Q_4^+) = (\mathcal{O}^+_{\rm VV+AA}, \mathcal{O}^+_{\rm SS+PP}, \mathcal{O}^+_{\rm VV-AA}, \mathcal{O}^+_{\rm SS-PP})$$

$$(\mathcal{Q}_1^+, \mathcal{Q}_2^+, \mathcal{Q}_3^+, \mathcal{Q}_4^+) = (\mathcal{O}^+_{\rm VA+AV}, \mathcal{O}^+_{\rm SP+PS}, \mathcal{O}^+_{\rm VA-AV}, \mathcal{O}^+_{\rm SP-PS})$$

Heavy quark spin symmetry

$$\psi_{\rm h} \to \exp(-i\phi_k\epsilon_{klm}\sigma_{lm})\psi_{\rm h}, \quad \bar{\psi}_{\rm h} \to \bar{\psi}_{\rm h}\exp(i\phi_k\epsilon_{klm}\sigma_{lm}),$$

$+ \; H(3)$ spatial rotations $+$ time reversal $\Rightarrow$ constrains on the renormalization matrix (at the non-perturbative level).

change of basis $(\vec{Q},\vec{\mathcal{Q}}) \to (\vec{Q}',\vec{\mathcal{Q}}')$ (with $Q'_1 \equiv Q_1$ and $\mathcal{Q}'_1 \equiv \mathcal{Q}_1$) $\Rightarrow$ parity-odd operators renormalise multiplicatively

$$\begin{pmatrix} \hat{\mathcal{Q}}'_1 \\ \hat{\mathcal{Q}}'_2 \\ \hat{\mathcal{Q}}'_4 \\ \hat{\mathcal{Q}}'_5 \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_1 & 0 & 0 & 0 \\ 0 & \mathcal{Z}_2 & 0 & 0 \\ 0 & 0 & \mathcal{Z}_4 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_5 \end{pmatrix} \begin{pmatrix} \mathcal{Q}'_1 \\ \mathcal{Q}'_2 \\ \mathcal{Q}'_4 \\ \mathcal{Q}'_5 \end{pmatrix}$$

$\Rightarrow$ use HQET for the b quark and twisted mass QCD for the light quarks. At $\alpha = \pi/2$ one can show that in the twisted basis, the parity-even operator of interests corresponds to the (chirally rotated) parity-odd one. For the renormalized matrix elements it holds:

$$\langle \bar{B}^0_q | \hat{Q}'^{+}_k(\mu) | B^0_q \rangle_{\mathrm{HQET}} = \mathcal{Z}'_k(g, a\mu) \langle \bar{B}^0_q | \mathcal{Q}'^{+}_k(a) | B^0_q \rangle^{\alpha=\pi/2}_{\mathrm{tmQCD}}$$

Using tmQCD, $\Delta F = 2$ and in particular $\Delta B = 2$ matrix elements can thus be computed with multiplicative renormalization.

We have computed the non-perturbative RCs $\mathcal{Z}'_k(g, a\mu)$ of the static HQET four-fermion operators in the SF scheme with $N_f = 0$ and $N_f = 2$ dynamical light quarks.

The computation requires the following ingredients:

1. perturbative calculation of the NLO anomalous dimension for the complete basis of static four-fermion operators in the SF scheme.

2. the non-perturbative running (step-scaling function $\sigma_k(u)$) in the continuum (in the SF scheme) for a wide range of couplings, i.e. on a wide range of scales. The SF allowed us to start at a $\mu_{\rm had} \approx 270\,\mathrm{MeV}$ and go up to $\mu_{\rm pt} \approx 70\,\mathrm{GeV}$.

   $$\sigma_k(u) = U_k(\mu, \mu/2) = \frac{\hat{c}_k(\mu/2)}{\hat{c}_k(\mu)} = \lim_{a\to 0} \frac{\mathcal{Z}_k(g_0, a\mu/2)}{\mathcal{Z}_k(g_0, a\mu)}\Big|_{u\equiv\bar{g}^2(\mu)}^{m=0}$$

   where $\quad \hat{c}_k(\mu) = \left[\frac{\bar{g}^2(\mu)}{4\pi}\right]^{-\gamma_k^{(0)}/2b_0} \exp\left\{-\int_0^{\bar{g}(\mu)} dg\left(\frac{\gamma_k(g)}{\beta(g)} - \frac{\gamma_k^{(0)}}{b_0 g}\right)\right\}$

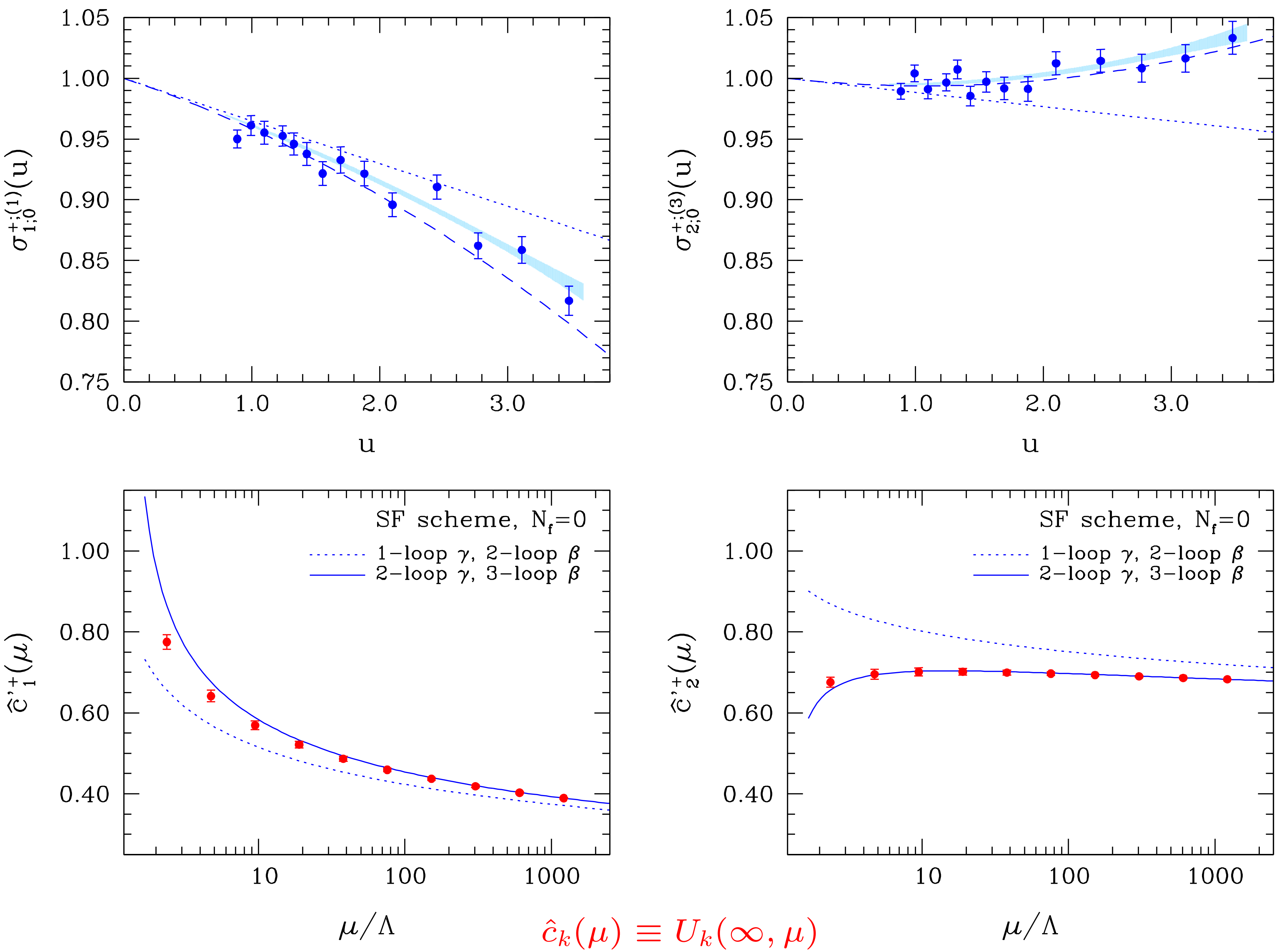
3. the matching factor $\mathcal{Z}_k(g_0, a\mu_{\rm had})$ at the low hadronic scale $\mu_{\rm had}$.

4. the RGI renormalization constant

   $$\hat{Z}_{k,\rm RGI}(g_0) = \hat{c}_k(\mu_{\rm had})\mathcal{Z}_k(g_0, a\mu_{\rm had}) = \hat{c}_k(\mu_{\rm pt})U_k(\mu_{\rm pt}, \mu_{\rm had})\mathcal{Z}_k(g_0, a\mu_{\rm had})$$

   with $\hat{c}_k(\mu_{\rm pt})$ perturbative at NLO while $U_k(\mu_{\rm pt}, \mu_{\rm had})$ non-perturbative.

Statistical + sytematic uncertainty $\leq 2\%$ for $N_f = 0$ and $\leq 5\%$ for $N_f = 2$.



$\hat{c}_k(\mu) \equiv U_k(\infty, \mu)$