Lattice QCD and Non-Perturbative Renormalization

GDR-Workshop

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INFN - TOR VERGATA

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Collaboration

Lecture 1:
the RI/MOM scheme

G.Martinelli, C. Pittori, C. Sachrajda, M.Testa, A.V., Nucl.Phys.B445 (1995) 81

Operator Renormalization

Renormalization and improvement

- the lattice formalism is a **bare** QFT
- computation results are **bare** WMEs at fixed UV cutoff $1/a$ (i.e. fixed $g_0^2(a)$)
- must renormalize them in order to get continuum physics
- UV cutoff is present at all stages of a computation and it is gradually increased

$$< f|\ Q_R(\mu)\ |i> = \lim_{a\to 0} \Big[\ Z_Q(a\mu, g_0^2)\ \ < f|\ Q(g_0^2)\ |i>\ + \mathcal{O}(a)\ \Big]$$

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bare WME depends on bare coupling and masses

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$$< f|\, Q_R(\mu)\, |i > \; = \; \lim_{a\to 0} \Big[\, Z_Q(a\mu, g_0^2) \; < f|\, Q(g_0^2)\, |i > \; + \; \mathcal{O}(a) \, \Big]$$

renormalized WME depends on dressed coupling, masses and scale

bare WME depends on bare coupling and masses

continuum limit obtained gradually by successive simulations

renorm. constant diverges logarithmically with a

discretization effects due to cutoff finiteness contaminate all computations

Renormalization and improvement

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- lattice renormalization can be done either in PT or non-perturbatively (NP)
- lattice PT is tedious and **badly convergent**; at say LO, it introduces large $\mathcal{O}(g_0^4)$ errors in Z_Q
- NP methods introduce $\mathcal{O}(a)$ **discretization errors** is Z_Q ; as also the bare WME has $\mathcal{O}(a)$ effects, this is preferable to PT
- better still: attempt to “help” continuum extrapolation by reducing all discretization errors to $\mathcal{O}(a^2)$ [Symanzik improvement; see later]

Renormalization and improvement

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- lattice renormalization can be done either in PT or non-perturbatively (NP)
- lattice PT is badly converging
  - example 1: MILC collaboration found that the strange quark mass was raised by 14% once its renormalization constant, known in 1-loop PT, was calculated at 2-loops
  - example 2: Göckeler et al. found that the strange quark mass was raised by 24% once its renormalization constant, known in 1-loop PT, was calculated by a NP method

Renormalization and improvement

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- lattice renormalization can be done either in PT or non-perturbatively (NP)
- lattice PT is badly converging
- two NP renormalization schemes have been devised for these renormalizations
    - RI/MOM scheme — G.Martinelli et al. Nucl.Phys.B445(1995)81
    - Schrödinger Functional (SF) scheme — M.Lüscher et al. Nucl.Phys.B478(1996)365

The RI/MOM Renormalization Scheme

G.Martinelli, C. Pittori, C. Sachrajda, M.Testa, A.V., Nucl.Phys.B445 (1995) 81

Generalities

- the scheme has a dual name: MOM stands for **momentum subtraction**. This is because the scheme mimics quite faithfully what is often done in continuum calculations

- the renormalization condition is imposed in momentum space.

- it consists in requiring that a given renormalized correlation function with, say, an operator insertion, is set, at fixed momenta μ, to its tree level value (a constant). This determines the operator renormalization constant, up to fundamental field renormalizations.

bare correlation functions

$$\begin{aligned} G_Q(x_1, \cdots, x_n) &\equiv < \phi(x_1) \cdots Q(0) \cdots \phi(x_n) > \\ \tilde{G}_Q(p) &= \mathcal{F}.\mathcal{T}.\ [G_Q(x)] \end{aligned}$$

renormalization condition

$$Z_\phi^n\left(\frac{\mu}{\Lambda}\right)\ Z_Q\left(\frac{\mu}{\Lambda}\right)\ \tilde{G}_Q(p)\Big|_{p^2=\mu^2} = \tilde{G}_Q^{(0)} = \text{const.}$$

- the renormalization condition is independent of the regularization scheme; thus we may adopt the lattice for the (**non-perturbative**) calculation of the bare correlation function in coordinate space, followed by its Fourier transform (a "discrete procedure" for a finite lattice) into momentum space.

- the scheme is **regularization independent** (RI), as opposed to the early days' lattice perturbation theory calculations, which relied on some continuum MS scheme.

Basic definitions

- consider a multiplicatively renormalizable operator

$$O^a_\Gamma(x) = \bar\psi(x)\Gamma\frac{\lambda^a}{2}\psi(x)$$

- examples: scalar/pseudoscalar densities etc.

$$S(x) = \bar\psi(x)\frac{\lambda^a}{2}\psi(x) \qquad P(x) = \bar\psi(x)\gamma_5\frac{\lambda^a}{2}\psi(x)$$

$$V_\mu(x) = \bar\psi(x)\gamma_\mu\frac{\lambda^a}{2}\psi(x) \qquad A_\mu(x) = \bar\psi(x)\gamma_\mu\gamma_5\frac{\lambda^a}{2}\psi(x)$$

- NB: we also consider vector and axial currents, which have finite normalizations
- NB: these operators are flavour non-singlets; e.g.

$$\begin{aligned} S(x) &= \bar u(x) d(x) \\ P(x) &= \bar s(x)\gamma_5 d(x) \end{aligned}$$

- we also need the quark propagator in momentum space (the integral is really a sum!!!)

$$\mathcal{S}(x_1 - x_2) = \langle\ \psi(x_1)\ \ \bar\psi(x_2)\ \rangle$$

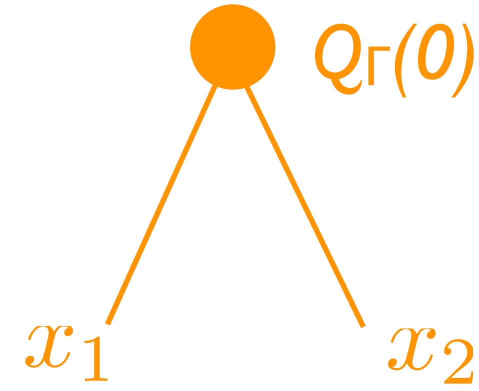
$$\mathcal{S}(p) \ = \ \int dx \ \ \exp(-ipx)\ \ \mathcal{S}(x)$$

- NB: easily computable on the lattice
- gauge fixing neccessary
- opt for Landau gauge

Basic definitions

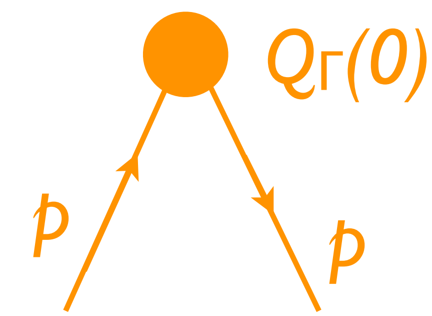
- the correlation function of interest, in coordinate space, is obtained by inserting the quark bilinear operator in 2-point fermionic Green function (the quark propagator)

$$G_Q(x_1, x_2) \;=\; \langle \psi(x_1)\; Q_\Gamma(0)\; \bar{\psi}(x_2) \rangle$$



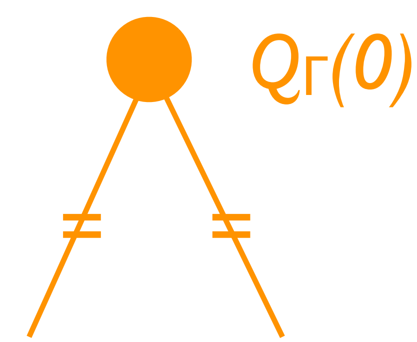
- Fourier tranform it to obtain the correlation function in momentum space

$$G_Q(p) \;=\; \int dx_1 dx_2 \;\exp(-ip[x_1 - x_2])\; G_Q(x_1, x_2)$$



- amputate the momentum space correlation function

$$\Lambda_Q(p) \;=\; \mathcal{S}^{-1}(p)\; G_Q(p)\; \mathcal{S}^{-1}(p)$$

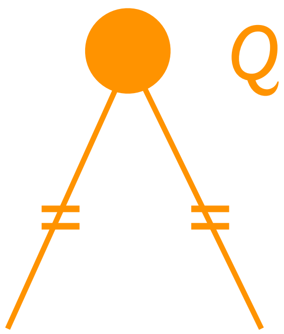


- NB: operator insertion has zero momentum transfer (optional)
- NB: all manipulations are in the Landau gauge
- the amputated correlation function is a matrix in Dirac-colour space; its tree level value is $\Gamma \otimes I$

Basic definitions

- it is convenient to impose the renormalization condition on a function of momenta (rather than on a Dirac-colour matrix)
- we thus "project" the amputated correlation Dirac-colour Green function by suitable traces
- this consists in defining the projected-amputated Green function

$$\Gamma_Q(p) \;\equiv\; \frac{1}{12}\mathrm{Tr}\left[\; P_Q \; \Lambda_Q(p) \;\right]$$



- the trace is over colour and spin indices
- the trace over colours is trivial
- the trace over spin is conditioned by the choice of the Dirac projectors P_Q, chosen so that the tree-level value of Γ_Q is unity (recall that the tree level value of is Γ_Q is $\Gamma \otimes I$).

$$P_S = I \qquad\qquad P_P = \gamma_5$$

$$P_V = \frac{1}{4}\gamma_\mu \qquad\qquad P_A = \frac{1}{4}\gamma_5\gamma_\mu$$

- NB: at tree level $\Gamma_Q = 1$

RI/MOM renormalization scheme

- so far we only defined a convenient projected-amputated correlation function $\Gamma_Q(p)$, in terms of the bilinear operator Q_Γ and the fermion fields ψ
- this bare quantity, regularized by the lattice, is computed non-perturbatively (i.e. numerically, at fixed UV cutoff)
- the renormalized $\Gamma_Q(p)$ is formally given by:

$$\Big[\Gamma_Q(p)\Big]_R = \lim_{a\to 0} \Big[\underbrace{Z_\psi^{-1}(a\mu)}_{\text{quark field renormalization}} \; \underbrace{Z_Q(a\mu)}_{\text{operator renormalization}} \; \Gamma_Q(p) \Big]$$

- RI/MOM renormalization scheme: impose the following renormalization condition on $\Gamma_Q(p)$

$$\Big[\Gamma_Q(p^2)\Big]_{\mathrm{R}}\Big|_{p^2=\mu^2} = Z_\psi^{-1}(a\mu) Z_Q(a\mu) \Gamma_Q(\mu) = 1$$

- i.e. the renormalized amputated-projected correlation function $[\Gamma_Q(p)]_R$, at scale μ, is set to its tree level vlue. From it the product Z_Q / Z_ψ is determined

RI/MOM renormalization scheme

$$\left[\Gamma_Q(p^2)\right]_{\mathrm{R}}\Big|_{p^2=\mu^2} = Z_\psi^{-1}(a\mu)Z_Q(a\mu)\Gamma_Q(\mu) = 1$$

- in practice the bare $\Gamma_Q(p)$ is computed at fixed UV cutoff (lattice spacing) for several quark masses m and renormalization scales μ
- being a mass-indeendent scheme, the chiral extrapolation m → 0 must be performed
- we must disentangle Z_Q from Z_ψ; conceptually the simplest way is by using the lattice conserved vector current V^C, which has $Z_{V^C} = 1$
- for this current, the RI/MOM condition gives a way to compute non-perturbatively Z_ψ

$$\left[\Gamma_{V^C}(p^2)\right]_{\mathrm{R}}\Big|_{p^2=\mu^2} = Z_\psi^{-1}(a\mu)\Gamma_{V^C}(\mu) = 1$$

- in practice this method is not applied because the conserved current is point split and somewhat intricate and costly to implement (in reality these are superable problems...)
- instead of V^C, one can use $Z_V V = V^C$, with Z_V taken from Ward identities
- people prefer to compute Z_ψ from the renormalization of the quark propagator

RI/MOM quark propagator renormalization

- the quark propagator is renormalized by the quark field renormalization parameters

$$\big[\mathcal{S}(p)\big]_{\mathrm{R}} \;=\; Z_\psi(a\mu)\;\mathcal{S}(p)$$

- the RI/MOM condition that fixes Z_ψ is given by

$$\frac{i}{12}\,\mathrm{Tr}\left[\frac{\not{p}\,\mathcal{S}_{\mathrm{R}}^{-1}}{p^2}\right]_{p^2=\mu^2} \;=\; Z_\psi^{-1}\,\frac{i}{12}\,\mathrm{Tr}\left[\frac{\not{p}\,\mathcal{S}^{-1}}{p^2}\right]_{p^2=\mu^2} \;=\; 1$$

- in practice the scheme is straightforward to apply: the non-perturbative computation of the quark propagator at a given lattice spacing and quark mass, on a gauge configuration ensemble, in the Landau gauge, is standard

- all other correlation functions we need, are constructed in terms of these propagators and include non-perturbative effects

- the scheme is defined in infinite volume; in practice we simulate at large volumes

- the results are the quantities $Z_Q(a\mu)$ and $Z_\psi(a\mu)$, computed for a large discrete set of scales $a\mu$

- recall that we must always extrapolate them to the chiral limit

RI/MOM window of applicability

- the range of renormalization scales, for which RI/MOM is applicable, is not unlimited
- there is a window in the range of aμ, for which the scheme works well

$$\Lambda_{\rm QCD} \;<<\; \mu \;<<\; \pi/a$$

this bound makes possible the matching with a perturbative scheme (e.g. MS) or with some Wilson coefficients of the OPE, calculated at large, perturbative scales aμ

this bound also potects the results from Goldstone pole contaminations (see below)

this bound satisfies the requirement of small cutoff effects $O(a\mu)$

- results are therefore expected to be "unreliable" at very small and very large aμ values
- in between we should be seing a more or less smooth signal
- it is better to combine the non-perturbative $Z_Q(a\mu)$ with the perturbative evolution function in order to compute Z_Q^{RGI} and check its independence from the scale aμ

RI/MOM window of applicability

$$Z_Q^{\mathrm{RGI}} = \left[\frac{\bar{g}^2(\mu)}{4\pi}\right]^{-\gamma_O^{(0)}/(2b_0)} \exp\left\{-\int_0^{\bar{g}(\mu)} dg\left(\frac{\gamma_O(g)}{\beta(g)} - \frac{\gamma_O^{(0)}}{b_0 g}\right)\right\} Z_Q(a\mu)$$

RGI renormalization parameter

scale evolution function, known in PT

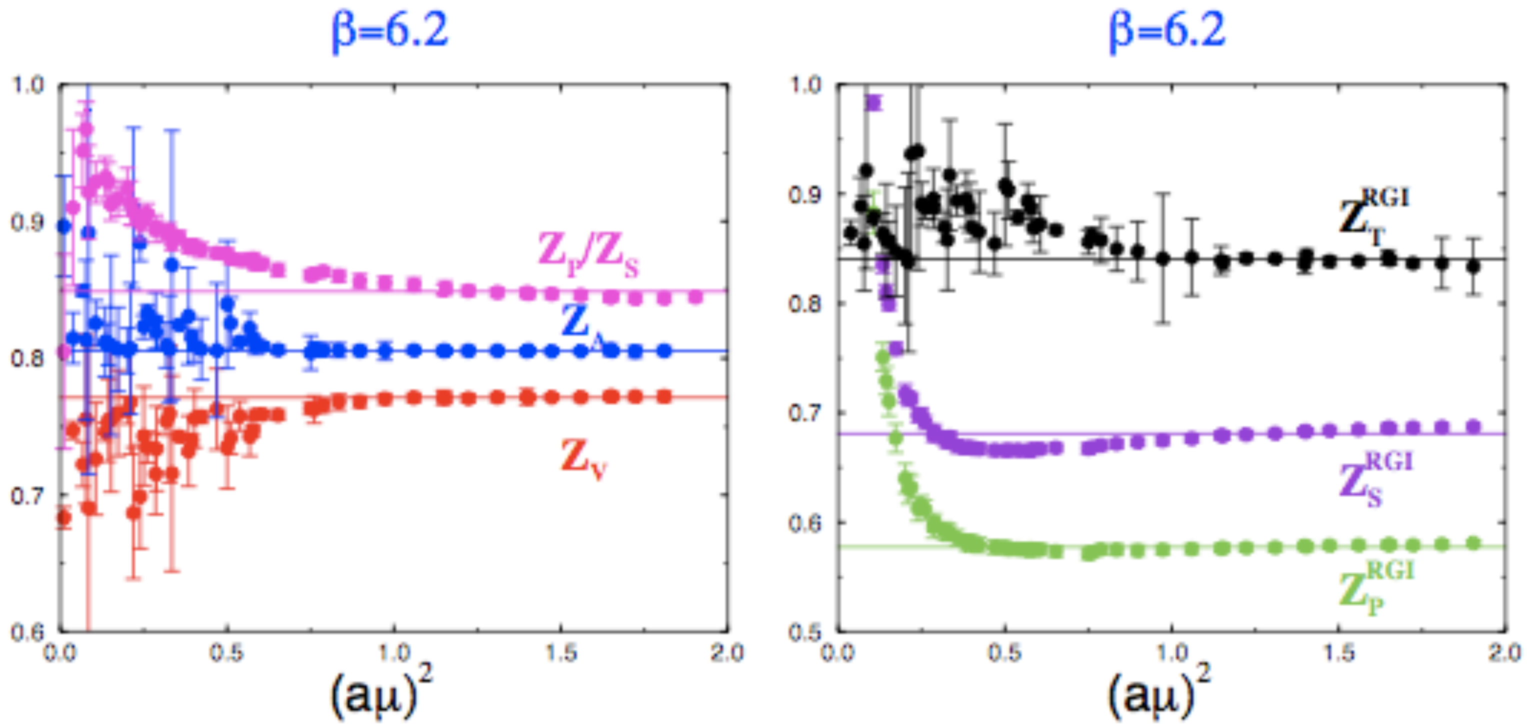
RI/MOM renormalization parameter, computed non-perturbatively

- the perturbative evolution function in the RI/MOM secheme is known to N^3LO for Z_S and Z_P and to N^2LO for Z_T

K.G.Chetyrkin, A. Retey, Nucl.Phys.B583 (2000) 3

- the gauge coupling needed in the above is taken from PT in $\overline{MS}$

RI/MOM window of applicability

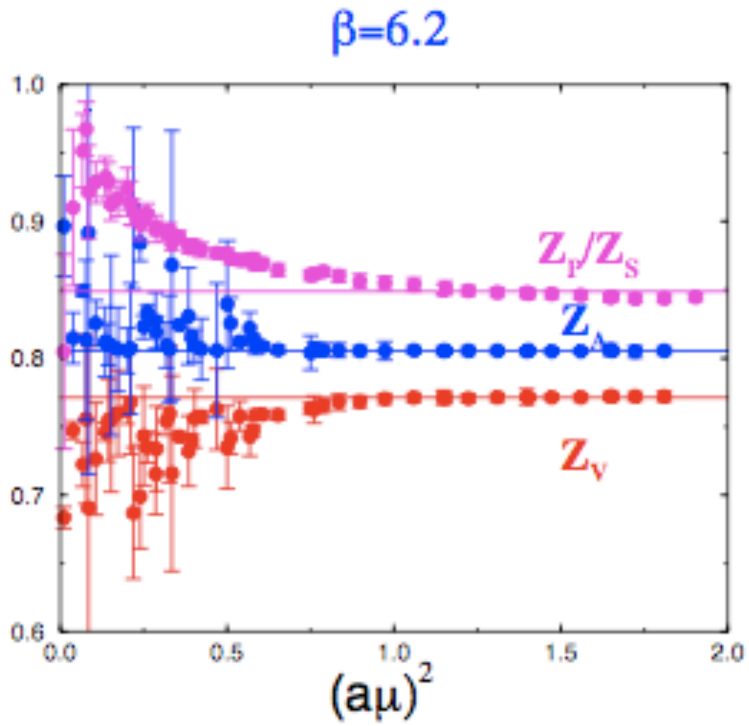


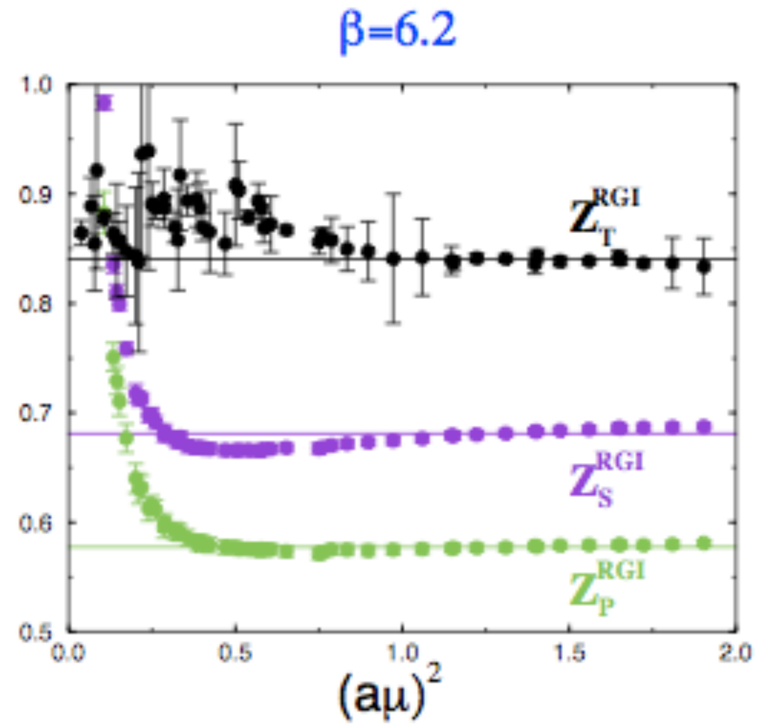
D.Becirevic et al., JHEP08 (2004) 022

- this is a quenched computation
- it is also *O*(a)-improved (Clover action etc.)

- Z_V^{RGI}, Z_A^{RGI} and Z_P^{RGI}/Z_S^{RGI} are scale independent as they should be
- the plateaux for Z_S^{RGI}, Z_P^{RGI}, and Z_T^{RGI} are reasonable; i.e. the perturbative subtraction of the scale dependence of the renormalization parameters works well
- this implies that PT appears to capture well the RG running (N.B. this statement is not valid in general for all RI/MOM renormalization parameters!)
- so in these plots the relevance of the non-perturbative calculation is in establishing the plateaux heights

RI/MOM window of applicability

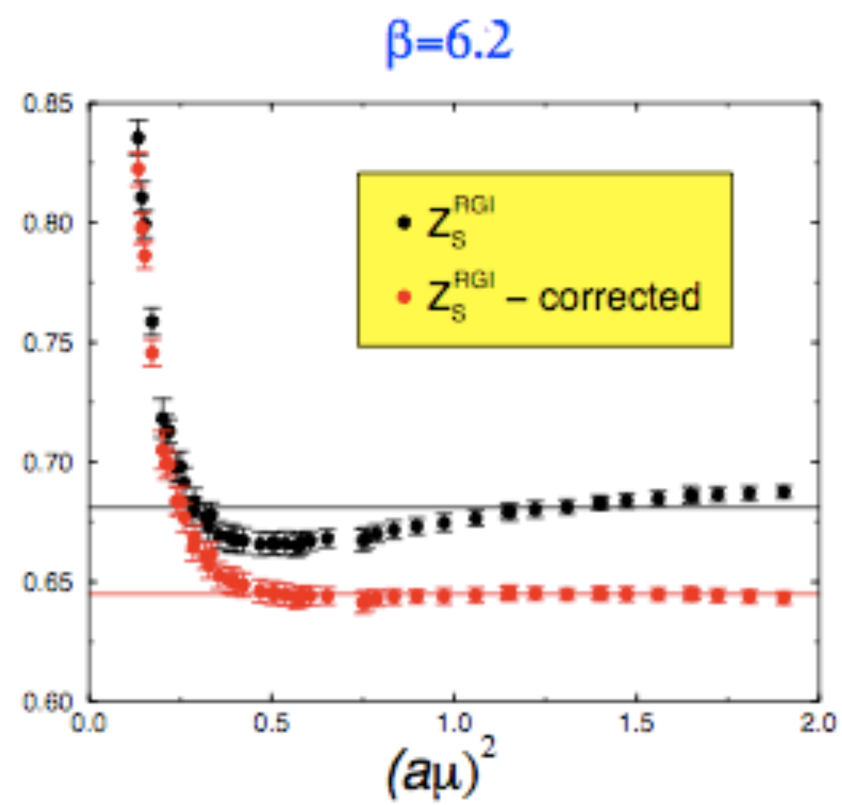


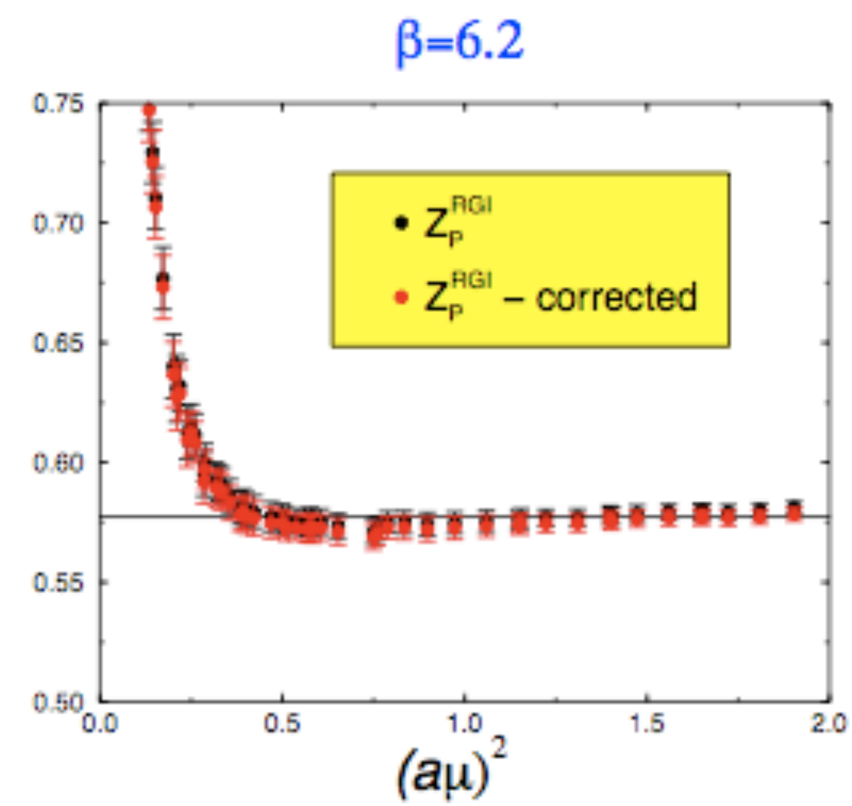


D.Becirevic et al., JHEP08 (2004) 022

- this is a quenched computation
- it is also O(a)-improved (Clover action etc.)

- the Z_S^{RGI} plateau is rather poor, compared to others
- subtract the leading pertrubative $O(g^2a^2)$ discretization effects
- it is not guaranteed that the plateaux will improve





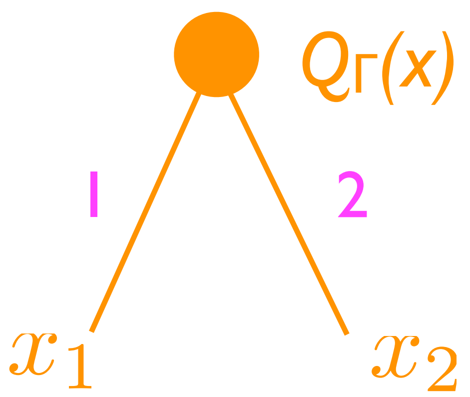
- 5% change for Z_S^{RGI}, 0% change for Z_P^{RGI}, 3% change for Z_T^{RGI}, 1% change for $Z_{V,A}^{RGI}$

RI/MOM compatibility with Ward identities

- scale independent current normalization constants (Z_V, Z_A) and scale independent ratios of operator renormalization constants (Z_S/Z_P) are fixed by Ward identities (WIs)
- they can also be determined through the RI/MOM condition
- are these two determinations compatible (up to discretization effects)?
- consider bilinear operators V_μ, A_μ, S, P made of two flavours ψ_1 and ψ_2 , with quark masses m_1 and m_2; i.e. bilinear operators Q are defined to be

$$Q_\Gamma = \bar{\psi}_1 \, \Gamma \, \psi_2$$

- WIs are now calculated for the operator insertions of the form



$$G_Q(x_1 - x, x_2 - x) = \langle \psi_1(x_1) \, Q_\Gamma(x) \, \bar{\psi}_2(x_2) \rangle$$

- in terms of such correlation functions, the vector WI (PCVC) is

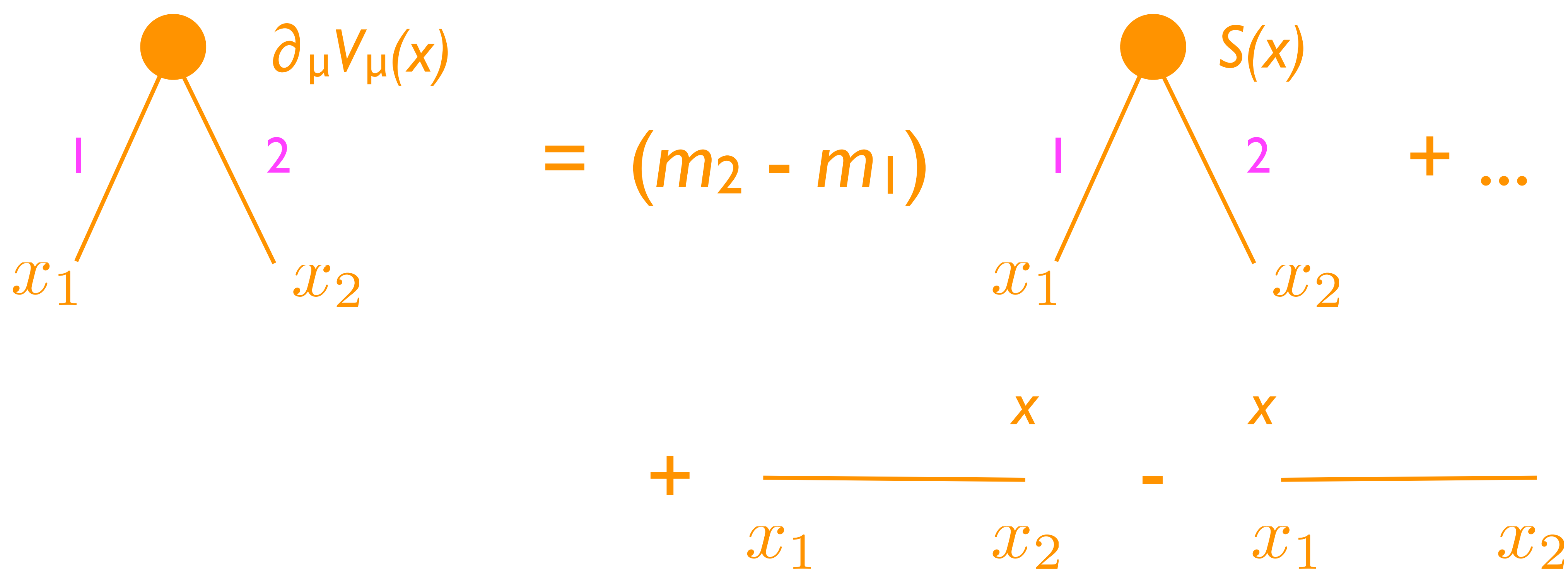
$$Z_V \sum_\mu \nabla_x^\mu G_V^\mu (x_1 - x, x_2 - x; m_1, m_2) = -(m_2 - m_1) \, G_S (x_1 - x, x_2 - x; m_1, m_2) + \delta(x_2 - x) \mathcal{S} (x_1 - x_2; m_1) - \delta(x_1 - x) \mathcal{S} (x_1 - x_2; m_2)$$

RI/MOM compatibility with Ward identities

- identification with formal PCVC

$$\boxed{\partial_\mu V_\mu} = \boxed{(m_1 - m_2)\ S} + \cdots$$

$$Z_V \sum_\mu \nabla_x^\mu G_V^\mu\,(x_1 - x, x_2 - x; m_1, m_2) \;=\; -(m_2 - m_1)\; G_S\,(x_1 - x, x_2 - x; m_1, m_2)$$
$$+\;\delta(x_2 - x)\mathcal{S}\,(x_1 - x_2; m_1) \;-\; \delta(x_1 - x)\mathcal{S}\,(x_1 - x_2; m_2)$$

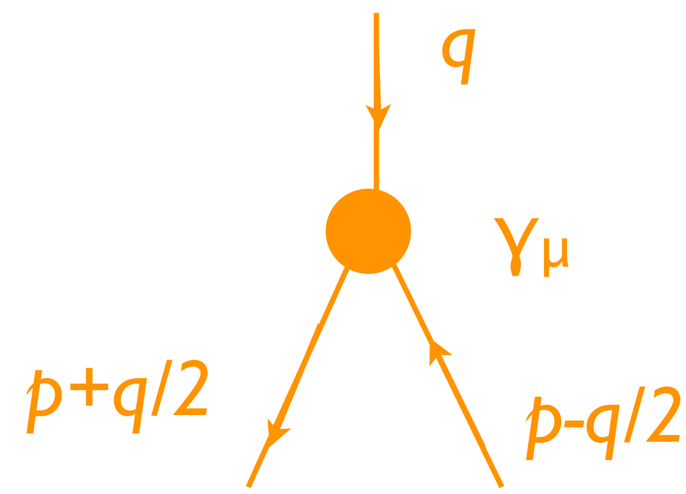


RI/MOM compatibility with Ward identities

$$Z_V \sum_\mu \nabla_x^\mu G_V^\mu \left(x_1 - x, x_2 - x; m_1, m_2\right) = -(m_2 - m_1)\, G_S \left(x_1 - x, x_2 - x; m_1, m_2\right) + \delta(x_2 - x)\mathcal{S}\left(x_1 - x_2; m_1\right) - \delta(x_1 - x)\mathcal{S}\left(x_1 - x_2; m_2\right)$$

- consider mass-degenerate case; this kills the correlation with the scalar operator
- Fourier transform above WI with distinct 4-momenta at each external leg (unlike RI/MOM)
- amputate (like in RI/MOM)

$$Z_V \sum_\mu i q_\mu \Lambda_V^\mu \left(p + \frac{q}{2}, p - \frac{q}{2}\right) = +\mathcal{S}^{-1}\left(p + \frac{q}{2}\right) - \mathcal{S}^{-1}\left(p - \frac{q}{2}\right)$$



- differentiate w.r.t. momentum transfer q_μ ; take limit $q_\mu \to 0$

$$Z_V \Lambda_V^\rho (p,p) + Z_V q_\mu \frac{\partial}{\partial q_\rho} \Lambda_V^\mu \left(p + \frac{q}{2}, p - \frac{q}{2}\right)\Bigg|_{q=0} = -i \frac{\partial}{\partial p_\rho} \mathcal{S}^{-1}(p)$$

vanishes in the limit $q_\mu \to 0$

RI/MOM compatibility with Ward identities

$$Z_V \Lambda_V^\rho(p,p) + Z_V q_\mu \frac{\partial}{\partial q_\rho} \Lambda_V^\mu\left(p+\frac{q}{2}, p-\frac{q}{2}\right)\Bigg|_{q=0} = -i\frac{\partial}{\partial p_\rho}\mathcal{S}^{-1}(p)$$

vanishes in the limit $q_\mu \to 0$

- project: multiply by γ_ρ ; take traces

$$Z_V \Gamma_V(p) = -\frac{i}{48}\mathrm{Tr}\left[\gamma_\rho \frac{\partial}{\partial p_\rho}\mathcal{S}^{-1}(p)\right]$$

- the RHS is Z_ψ ; thus the PCVC has been shown equivalent to the RI/MOM condition!

RI/MOM compatibility with Ward identities

- recall that a vector WI (PCVC) is:

$$Z_V \Lambda_V^\rho (p,p) + Z_V q_\mu \frac{\partial}{\partial q_\rho} \Lambda_V^\mu \left(p + \frac{q}{2}, p - \frac{q}{2}\right)\Bigg|_{q=0} = -i \frac{\partial}{\partial p_\rho} \mathcal{S}^{-1}(p)$$

- performing similar steps (in **maselss** case) we find the axial WI (PCAC):

$$Z_A \Lambda_A (p,p) + Z_A q_\mu \frac{\partial}{\partial q_\rho} \Lambda_A \left(p + \frac{q}{2}, p - \frac{q}{2}\right)\Bigg|_{q=0} = -\frac{i}{2}\left[\gamma_5 \frac{\partial}{\partial p_\rho} \mathcal{S}^{-1}(p) - \frac{\partial}{\partial p_\rho} \mathcal{S}^{-1}(p)\gamma_5\right]$$

- does **NOT vanish** in the limit $q_\mu \to 0$, due to the presence of a massless **Goldstone boson**. This term is needed in order to saturate the PCAC correctly.

- it **DOES vanish** in the limit $p^2 \to \infty$, as shown by an OPE argument

G.Martinelli, C. Pittori, C. Sachrajda, M.Testa, A.V., Nucl.Phys.B445 (1995) 81

- thus in the limit $p^2 \to \infty$, we recover (just like vector case) that the RI/MOM determination of Z_A is compatible with PCAC

- we must still address the issue of whether our momenta are high enough

RI/MOM compatibility with Ward identities

- how about the Z_S/Z_P ratio? It is also fixed by WIs; start again with the PCVC relation

$$Z_V \sum_\mu \nabla_x^\mu G_V^\mu (x_1 - x, x_2 - x; m_1, m_2) = -(m_2 - m_1)\, G_S (x_1 - x, x_2 - x; m_1, m_2) + \delta(x_2 - x)\mathcal{S}(x_1 - x_2; m_1) - \delta(x_1 - x)\mathcal{S}(x_1 - x_2; m_2)$$

- this time integrate over all space (for massive quarks!!!)
- integration kills the vector current term (LHS), as it is a surface term

$$(m_2 - m_1) \int d^4x\ G_S (x_1 - x, x_2 - x) = \mathcal{S}(x_1 - x_2; m_1) - \mathcal{S}(x_1 - x_2; m_2)$$

- Fourier transform, amputate, project...

$$(m_2 - m_1)\Gamma_S (p; m_1, m_2) = -\frac{1}{12}\mathrm{Tr}\Big[\mathcal{S}^{-1}(p; m_1)\Big] + \frac{1}{12}\mathrm{Tr}\Big[\mathcal{S}^{-1}(p; m_2)\Big]$$

- in the mass degenerate limit

$$\Gamma_S (p) = \frac{1}{12}\mathrm{Tr}\left[\frac{\partial \mathcal{S}^{-1}(p; m_2)}{\partial m}\right]$$

RI/MOM compatibility with Ward identities

- recap: for the PCVC case we find

$$(m_2 - m_1)\Gamma_S\,(p; m_1, m_2) = -\frac{1}{12}\mathrm{Tr}\Big[\mathcal{S}^{-1}\,(p; m_1)\Big] + \frac{1}{12}\mathrm{Tr}\Big[\mathcal{S}^{-1}\,(p; m_2)\Big]$$

  - in the mass degenerate limit $\qquad \Gamma_S\,(p) = \frac{1}{12}\mathrm{Tr}\left[\frac{\partial \mathcal{S}^{-1}\,(p; m)}{\partial m}\right]$

- analogously for the PCAC case we find

$$(m_1 + m_2)_{\mathrm{PCAC}}\;\Gamma_P\,(p) = \frac{1}{12}\mathrm{Tr}\Big[\mathcal{S}^{-1}\,(p; m_1)\Big] + \frac{1}{12}\mathrm{Tr}\Big[\mathcal{S}^{-1}\,(p; m_2)\Big]$$

- in the mass degenerate limit, differentiating w.r.t. the PCAC mass

$$\Gamma_P\,(p) \;+\; m_{\mathrm{PCAC}}\frac{\partial \Gamma_P}{\partial m_{\mathrm{PCAC}}} \;=\; \frac{1}{12}\mathrm{Tr}\left[\frac{\partial \mathcal{S}^{-1}\,(p; m_{\mathrm{PCAC}})}{\partial m_{\mathrm{PCAC}}}\right]$$

RI/MOM compatibility with Ward identities

- vector WI: $$\Gamma_S(p) = \frac{1}{12}\mathrm{Tr}\left[\frac{\partial \mathcal{S}^{-1}(p;m)}{\partial m}\right]$$
- axial WI: $$\Gamma_P(p) \; + \; m_{\mathrm{PCAC}}\frac{\partial \Gamma_P}{\partial m_{\mathrm{PCAC}}} \; = \; \frac{1}{12}\mathrm{Tr}\left[\frac{\partial \mathcal{S}^{-1}(p;m_{\mathrm{PCAC}})}{\partial m_{\mathrm{PCAC}}}\right]$$
- we express these WIs in terms of renormalized quantities: $$m \; = \; m_0 - m_{\mathrm{cr}}$$

$$P_R \; = \; Z_P \; P \qquad S_R \; = \; Z_S \; S \qquad m_R \; = \; Z_S^{-1} \; m \; = \; Z_P^{-1} \; m_{\mathrm{PCAC}}$$

- all this combines to a WI determination of the ratio Z_S/Z_P $$\frac{Z_P}{Z_S} \; = \; \frac{\frac{\Gamma_S}{\Gamma_P}}{1 \; + \; \frac{m}{\Gamma_P}\frac{\partial \Gamma_P}{\partial m}}$$
- the RI/MOM determination is $$\frac{Z_P}{Z_S} \; = \; \frac{\Gamma_S}{\Gamma_P}$$
- the two determinations differ by a factor which becomes negligible in the limit $p^2 \rightarrow \infty$ (see below)
- the absence of this factor from the RI/MOM determination consists of a **Goldstone pole contamination**

RI/MOM and Goldstone pole contamination

- the Goldstone pole contribution is seen more explicitly in the non-perturbative part of the quark propagator (obtained from an OPE argument)

$$\mathcal{S}^{-1}(p;m) \;=\; i\not{p}\,\Sigma_1(p^2;m;\mu^2) + m\,\Sigma_2(p^2;m;\mu^2) \;+\; \Sigma_3(p;m;\mu^2)$$

perturbative form factors

non-perturbative form factor

$$\Sigma_3 \;=\; K\; g^2\,\frac{<\bar{\psi}\psi>}{p^2} \;+\; \mathcal{O}(p^{-4})$$

P. Pasqual & E. deRafael Z.Phys.C12 (1982) 127

- plug this in the WI relations between Γ_S (and Γ_P) and the quark propagator
  - vector WI:
  - axial WI:

$$\Gamma_S\,(p) = \frac{1}{12}\mathrm{Tr}\left[\frac{\partial\mathcal{S}^{-1}\,(p;m_2)}{\partial m}\right]$$

$$\Gamma_P\,(p) \;+\; m_{\mathrm{PCAC}}\frac{\partial\Gamma_P}{\partial m_{\mathrm{PCAC}}} \;=\; \frac{1}{12}\mathrm{Tr}\left[\frac{\partial\mathcal{S}^{-1}\,(p;m_{\mathrm{PCAC}})}{\partial m_{\mathrm{PCAC}}}\right]$$

RI/MOM and Goldstone pole contamination

- the Goldstone pole contribution is seen more explicitly in the non-perturbative part of the quark propagator

$$\Gamma_S(p;m_1,m_2) \;=\; \Sigma_2(p;m) + m\frac{\partial\Sigma_2(p;m)}{\partial m} + \mathcal{O}\left(p^{-4}\right)$$

perturbative contribution

NLO non-perturbative contribution

LO non-perturbative contribution

$$\Gamma_P(p;m) = \frac{Z_S}{Z_P}\Sigma_2\left(ap,am\right) - \frac{Z_\psi Z_S}{m}K' g_0^2 \frac{\langle\bar{\psi}\psi\rangle_s}{p^2}$$

- to LO in $1/p$, Γ_S is free of non-perturbative contributions, while is Γ_P not
- the Γ_P non perturbative contribution diverges in the chiral limit (divergence is of IR type) !!!
- the Γ_P non perturbative contribution vanishes at large momenta

RI/MOM and Goldstone pole contamination

- the Goldstone pole contribution is seen more explicitly in the non-perturbative part of the quark propagator

$$\Gamma_S(p;m_1,m_2) = \Sigma_2(p;m) + m\frac{\partial\Sigma_2(p;m)}{\partial m} + \mathcal{O}\left(p^{-4}\right)$$

perturbative contribution

NLO non-perturbative contribution

LO non-perturbative contribution

$$\Gamma_P(p;m) = \frac{Z_S}{Z_P}\Sigma_2\left(ap,am\right) - \frac{Z_\psi Z_S}{m}K' g_0^2\frac{\langle\bar{\psi}\psi\rangle_s}{p^2}$$

- the WI determination of Z_S/Z_P has a denominator term (absent from the RI/MOM determination) which eliminates the divergent behaviour of Γ_P at vanishing quark mass

$$\frac{Z_P}{Z_S} = \frac{\frac{\Gamma_S}{\Gamma_P}}{1+\frac{m}{\Gamma_P}\frac{\partial\Gamma_P}{\partial m}}$$

- on the contrary, the RI/MOM determination has this Goldstone pole contamination

RI/MOM and Goldstone pole contamination

- how much is our data affected by this?
- the contamination even at large ("pertrubative"?) scales of about 1-2 GeV has been recognized in early data

J.R.Cudell, A. Le Yaouanc, C. Pittori, Phys.Lett.B454 (1999) 105
J.R.Cudell, A. Le Yaouanc, C. Pittori, Phys.Lett.B516 (2001) 92

- the proposal of the above authors was to redefine Z_P by identifying, fitting and removing the $1/p^2$ behaviour from the Γ_P data, at finite quark mass m
- the corrected data is then extrapolated to zero quark mass
- another way is to implement the following combination of vector and axial WIs in the computation of Z_S/Z_P; note that non-degenerate masses are ivolved

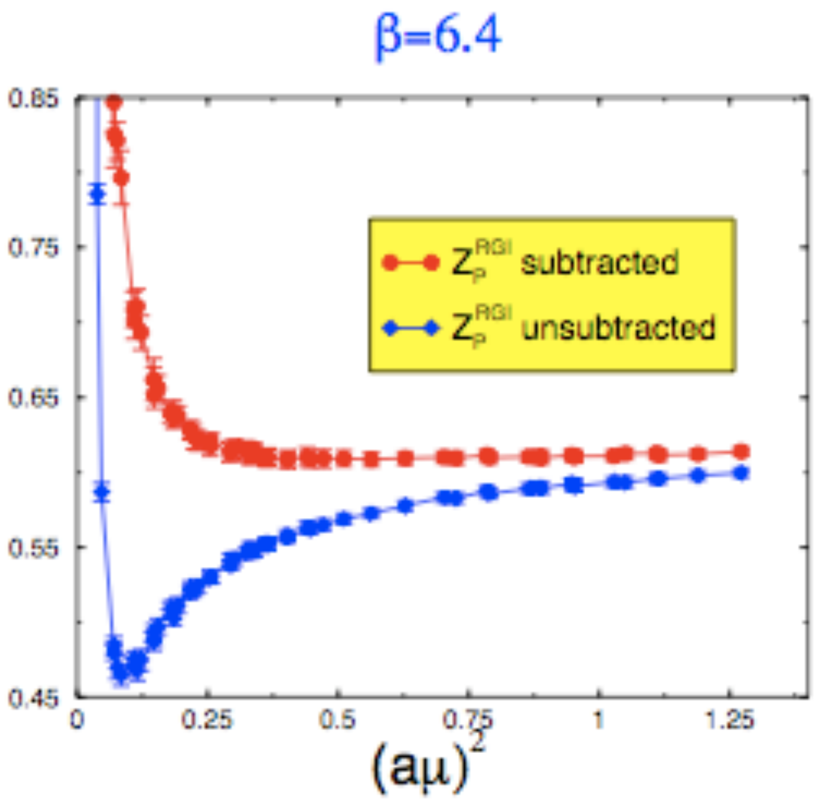
$$\frac{Z_P}{Z_S} = \frac{(m_1 - m_2)\,\Gamma_S\,(p; m_1, m_2)}{m_1 \Gamma_P\,(p; m_1) - m_2 \Gamma_P\,(p; m_2)}$$

- the non-pertrurbative contributions in the denominator cancel at LO in $1/p^2$

$$\Gamma_P(p;m) = \frac{Z_S}{Z_P}\Sigma_2\,(ap, am) - \frac{Z_\psi Z_S}{m} K' g_0^2 \frac{\langle\bar\psi\psi\rangle_s}{p^2}$$

L.Giusti, A.V., Phys.Lett.B488 (2000) 303

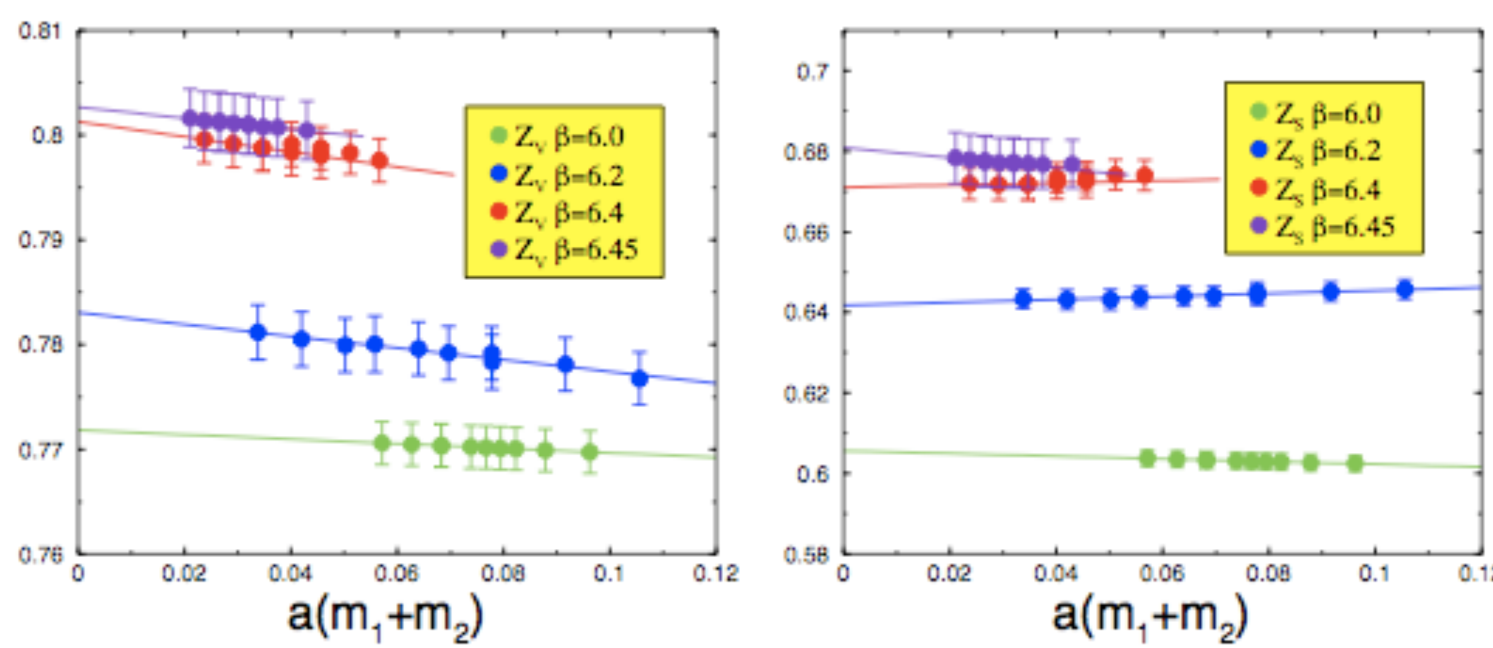
RI/MOM and Goldstone pole contamination



D.Becirevic et al., JHEP08 (2004) 022

- this is a quenched computation
- the low p^2 behaviour is significantly modified (smoothed out)
- this results to a better plateau at large p^2 (but datasets appear to be converging)

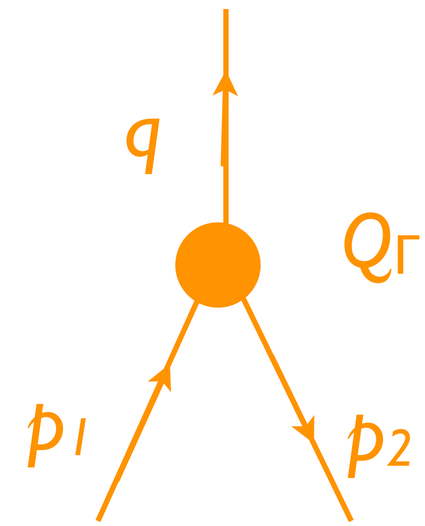
- extrapolate linearly to vanishing quark mass



RI/MOM recapitulation

- compute *Z*'s from amputated - projected correlation functions at fixed coupling and many momentum scales
- divide out the discretization effects predicted by lowest order perturbation theory (optional)
- correct for Goldstone pole contaminations wherever they appear
- extrapolate to vanishing quark mass
- NB: personal prejudice: when applicable (Z_V, Z_A and Z_P/Z_S), prefer WIs (explicitly scale independent) to RI/MOM; in this way you avoid a systrematic uncertianty due to the Goldstone pole
- once you have from WIs, compute Z_P as the product $[\ Z_P/Z_S\ ] \times Z_S$, thus avoiding problems with Goldstone pole contaminations

- NEW proposal is the RI/SMOM scheme: work with non-exeptional momenta $p_1^2 = p_1^2 = q^2$
- this removes the dominant Goldstone pole effect



Y. Aoki et al., Phys.Rev. D78 (2008) 054510
Y. Aoki, PoS LATTICE 2008 (2008) 222