Lattice QCD and Non-Perturbative Renormalization

GDR-Workshop

A. Vladikas
INFN - TOR VERGATA

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Lecture 2:
Schrödinger Functional

RG-running on the lattice: motivation

Operator RG-running

- suppose a quantity $Q(\mu)$ (quark mass, operator WME) is renormalized in a NP scheme

$$Q_R(\mu) \;=\; \lim_{a\to 0}\; Z_Q(g_0^2,\, a\mu)\; Q(g_0^2)$$

- if you use a hadronic scheme, the renormalization scale is going to be low $\mu \sim m_H$
- you need to know $Q(\mu)$ at a larger scale either for conventional reasons (e.g. people are used to MS-scheme quark masses $m_q(\mu)$ with $\mu \sim 2$GeV) or for matching with perturbative scales, as in the OPE:

$$Q^{\rm phys} \;=\; \sum\; C_W(\mu)\; \lim_{a\to 0}\; \left[\; Z_Q(g_0^2, a\mu)\;\; < f|\, Q(g_0^2)\, |i> \;\right]$$

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Wilson coefficients
calculated in PT
short-distance effects

renormalization scale
must be large; say 10GeV

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must be O(1), so as to avoid large logs
must be smaller than 1, so as to avoid discretization errors

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- if we wish to compute everything at one go (a single lattice) we must also ensure that $m_H L >> 1$, in order to avoid finite size errors
- i.e. we must satisfy $L >> 1/m_H \sim 1/(0.15\ \mathrm{GeV}) >> 1/\mu \sim 1/(10\ \mathrm{GeV}) > a$
- IMPOSSIBLE on present day resources as it gives $L/a = O(100\text{-}1000)$

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- need to compute the renormalized WME at a hadronic (low) scale μ_{min} and then do RG-running all the way to a perturbative (high) scale μ_{max}

- an option is using PT for the RG running, introducing ill-controlled $O(g^n)$ systematic errors

- the SF scheme, combined with finite size techniques, is the only one used so far for non-perturbative RG-running

RG-running: generalities

RG-running in the continuum

- the basic idea is always that of Callan-Symanzik
- there are mass-independent renormalization schemes, in which the renormalization conditions are imposed at the chiral limit (this is sufficient to remove UV divergences)
- in such schemes the renormalization constants and running functions do not depend on the theory's masses: $Z_g(a\mu, g_0)$, $Z_m(a\mu, g_0)$, $\beta(g_R)$, $\gamma(g_R)$ etc.
- first we reformulate what we know from continuum QCD renormalization (usually worked out in PT) in a general, non-perturbative (N.P.) language, suitable to N.P. computations
- we start with the RG-running of the gauge coupling, expressed in terms of the Callan-Symanzik β-function

$$\beta(g_{\mathrm{R}}) \;=\; \mu \, \frac{\partial g_{\mathrm{R}}}{\partial \mu}$$

- it is simple to integrate this from a reference scale μ_0 to a general scale μ

$$\frac{\mu_0}{\mu} \;=\; \exp\Big[- \int_{g_{\mathrm{R}}(\mu_0)}^{g_{\mathrm{R}}(\mu)} \frac{dg}{\beta(g)} \Big]$$

RG-running in the continuum

$$\frac{\mu_0}{\mu} = \exp\left[- \int_{g_R(\mu_0)}^{g_R(\mu)} \frac{dg}{\beta(g)} \right]$$

- it is natural, for an asymptotically free theory (QCD), to choose the reference scale $\mu_0 \to \infty$, for which $g_R(\mu_0) \to 0$
- we know, however, the perturbative behaviour of the beta function at small couplings

$$\beta(g) = -g^3 \left[b_0 + b_1\, g^2 + b_2\, g^4 + \cdots \right]$$

$$b_0 = \frac{1}{(4\pi)^2}\left[11 - \frac{2N_f}{3}\right]$$

$$b_1 = \frac{1}{(4\pi)^4}\left[102 - \frac{38N_f}{3}\right]$$

universal

renormalization scheme dependent

- the perturbative expression for $\beta(g_R)$ tells us that the above integral diverges at the lower end $g_R(\mu_0) = 0$, due to the first two terms of the expansion (NLO)

RG-running in the continuum

$$\frac{\mu_0}{\mu} = \exp\Big[-\int_{g_R(\mu_0)}^{g_R(\mu)} \frac{dg}{\beta(g)}\Big]$$

- trick: add and subtract the potentially diverging term $1/\beta_{NLO}(g_R)$ tin the intergrarnd:

$$\mu_0 = \mu \exp\Big[-\underbrace{\int_{g_R(\mu_0)}^{g_R(\mu)} dg \, \big[\frac{1}{\beta(g)} - \frac{1}{\beta_{NLO}(g)}\big]}_{\text{regular in the limit } g_R(\mu_0) \to 0}\Big] \; \exp\Big[-\underbrace{\int_{g_R(\mu_0)}^{g_R(\mu)} dg \frac{1}{\beta_{NLO}(g)}}_{\text{divergent in the limit } g_R(\mu_0) \to 0\text{; calculable for } g_R(\mu_0) \neq 0}\Big]$$

- calculate the NLO integral (for $g_R(\mu_0) \neq 0$) and carry everything that depends on μ_0 to the LHS, leaving all μ-dependent quantities on the RHS

$$\mu_0 \exp\Big[-\frac{1}{2b_0 g_R^2(\mu_0)}\Big] \Big[b_0 g_R^2(\mu_0)\Big]^{-b_1/(2b_0^2)} =$$

$$\mu \exp\Big[-\frac{1}{2b_0 g_R^2(\mu)}\Big] \Big[b_0 g_R^2(\mu)\Big]^{-b_1/(2b_0^2)} \exp\Big[-\int_{g_R(\mu_0)}^{g_R(\mu)} dg \, \big[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\big]\Big]$$

RG-running in the continuum

$$\mu_0 \ \exp\Big[-\frac{1}{2b_0 g_{\rm R}^2(\mu_0)}\Big]\ \Big[b_0 g_{\rm R}^2(\mu_0)\Big]^{-b_1/(2b_0^2)} =$$

$$\mu\ \exp\Big[-\frac{1}{2b_0 g_{\rm R}^2(\mu)}\Big]\ \Big[b_0 g_{\rm R}^2(\mu)\Big]^{-b_1/(2b_0^2)} \exp\Big[-\int_{g_{\rm R}(\mu_0)}^{g_{\rm R}(\mu)} dg\ \big[\frac{1}{\beta(g)}+\frac{1}{b_0 g^3}-\frac{b_1}{b_0^2 g}\big]\Big]$$

- in the limit $g_R(\mu_0) \to 0$, the RHS is μ_0 independent; therefore the same holds for the LHS
- this enables us to define an energy scale, typical of the theory

$$\Lambda_{\rm QCD} \ \equiv\ \lim_{\mu_0\to\infty} \mu_0\ \exp\Big[-\frac{1}{2b_0 g_{\rm R}^2(\mu_0)}\Big]\ \Big[b_0 g_{\rm R}^2(\mu_0)\Big]^{-b_1/(2b_0^2)}$$

$$\Lambda_{\rm QCD} \ =\ \mu\ \exp\Big[-\frac{1}{2b_0 g_{\rm R}^2(\mu)}\Big]\ \Big[b_0 g_{\rm R}^2(\mu)\Big]^{-b_1/(2b_0^2)} \exp\Big[-\int_{0}^{g_{\rm R}(\mu)} dg\ \big[\frac{1}{\beta(g)}+\frac{1}{b_0 g^3}-\frac{b_1}{b_0^2 g}\big]\Big]$$

RG-running in the continuum

$$\Lambda_{\rm QCD} \;\equiv\; \lim_{\mu_0\to\infty} \mu_0 \;\exp\left[-\frac{1}{2b_0 g_{\rm R}^2(\mu_0)}\right]\;\left[b_0 g_{\rm R}^2(\mu_0)\right]^{-b_1/(2b_0^2)}$$

$$\Lambda_{\rm QCD} \;=\; \mu\;\exp\left[-\frac{1}{2b_0 g_{\rm R}^2(\mu)}\right]\;\left[b_0 g_{\rm R}^2(\mu)\right]^{-b_1/(2b_0^2)} \exp\left[-\int_0^{g_{\rm R}(\mu)} dg\;\left[\frac{1}{\beta(g)}+\frac{1}{b_0 g^3}-\frac{b_1}{b_0^2 g}\right]\right]$$

- this is an exact expression, from which standard PT results for LO and NLO cases may be obtained
- the "miracle" of renormalization is that, even for massless QCD, it generates an energy scale
- $\Lambda_{\rm QCD}$ is Renormalization Group Invariant (RGI; i.e. μ-independent) but depends on the renormalization scheme (β is scheme independent only to NLO order)
- $\Lambda_{\rm QCD}$ depends on the number of quark flavours (cf. b_0, b_1) but not on the value of the quark masses; in fact it may be calculated in PT, or computed NP^{ly} with N_f massless quarks
- already at LO you can see from above that $\Lambda_{\rm QCD}$ corresponds to a NP coupling (oxymoron!)

$$g_{\rm R}^2(\mu) \;=\; -\frac{1}{2b_0 \ln(\mu/\Lambda_{\rm QCD})}$$

RG-running in the continuum

$$\Lambda_{\mathrm{QCD}} = \mu \, \exp\Big[- \frac{1}{2b_0 g_{\mathrm{R}}^2(\mu)}\Big] \, \Big[b_0 g_{\mathrm{R}}^2(\mu)\Big]^{-b_1/(2b_0^2)} \exp\Big[- \int_0^{g_{\mathrm{R}}(\mu)} dg \, \big[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\big]\Big]$$

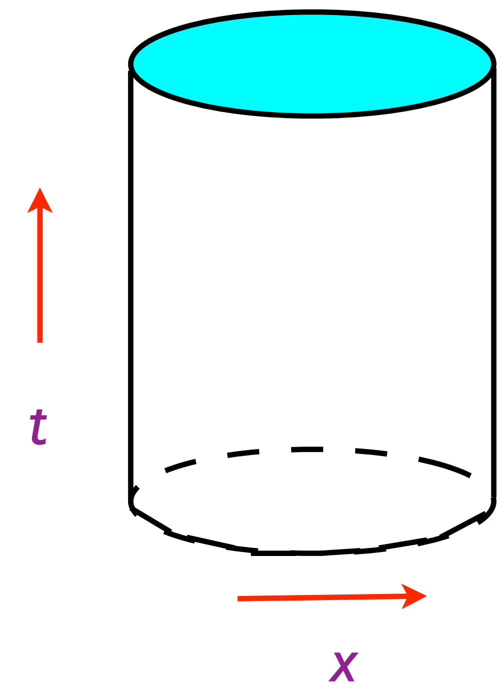
- suppose we have chosen a scheme; i.e. we have a definition of $g_R(\mu)$, accompanied by a renormalization condition for the coupling
- suppose that we have developed a powerful NP method (lattice) with which to compute $\beta(\mu)$ in a vast range of scales: from, say $\mu_{min} \sim \Lambda_{QCD}$ to $\mu_{max} \sim 100$ GeV
- the above tells us that the dimensionless ratio Λ_{QCD}/μ can be calculated from first principles of QCD, without any "physical" input (e.g. a hadronic mass or any other experimentally known quantity); this ratio is a "pure" Quantum Field Theory quantity
- a "physical" input is required (as shown below) in order to establish the correspondence of a given reference coupling $g_R(\mu_{ref})$ to its scale μ_{ref} (in GeV); from this, Λ_{QCD} (in GeV) is immediately obtained
- we will show that the Schrödinger Functional renormalization scheme beautifully fulfills these expectations

The Schrödinger Functional

M.Lüscher, R.Narayanan, P.Weisz, U.Wolff Nucl.Phys.B384(1992)168
M.Lüscher, R.Sommer, U.Wolff, P.Weisz Nucl.Phys.B389(1993)247
S. Sint Nucl.Phys.B421(1994)135; Nucl.Phys.B451(1995)416
M.Lüscher, S.Sint, R.Sommer, P.Weisz Nucl.Phys.B478(1996)365
S.Capitani, M.Lüscher, R.Sommer, H.Wittig Nucl.Phys.B544(1999)669

Schrödinger Functional in the continuum

- the SF scheme is defined in a finite L^4 volume, with periodic boundary conditions (b.c.'s) in space and Dirichlet b.c.'s in time
- For a Yang-Mills theory this means that we must specify the gauge configurations at the time boundaries



$$A_\mu^\Omega(x) \;=\; \Omega(x)A_\mu(x)\Omega^{-1}(x) \;+\; \Omega(x)\partial_\mu\Omega(x)^{-1}$$

Gauge field

Gauge transformation

$$A_k(x) = C_k^\Omega(\vec{x}) \qquad @ \;\; x_0 = 0$$
$$A_k(x) = C_k'(\vec{x}) \qquad @ \;\; x_0 = L$$

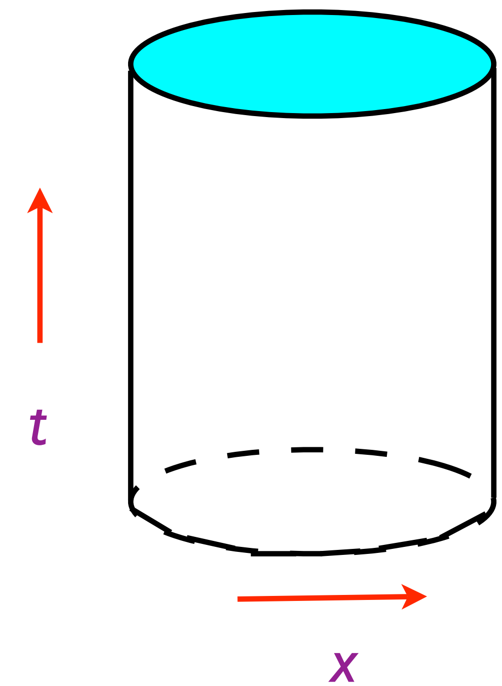
Dirichlet b.c.'s at time boundaries

periodic b.c.'s in space

$$A_k(x) \;=\; A_k(x + L\vec{k})$$
$$\Omega(\vec{x}) \;=\; \Omega(\vec{x} + L\vec{k}) \qquad @ \; x = (\vec{x}, 0)$$

Schrödinger Functional in the continuum

- the SF scheme is defined in a finite L^4 volume, with periodic boundary conditions (b.c.'s) in space and Dirichlet b.c.'s in time
- the Euclidean partition function defines the SF



$$\mathcal{Z}[C', C] \ = \ \int \mathcal{D}[\Omega] \int \mathcal{D}[A_\mu] \exp\{-S_G[A]\}$$

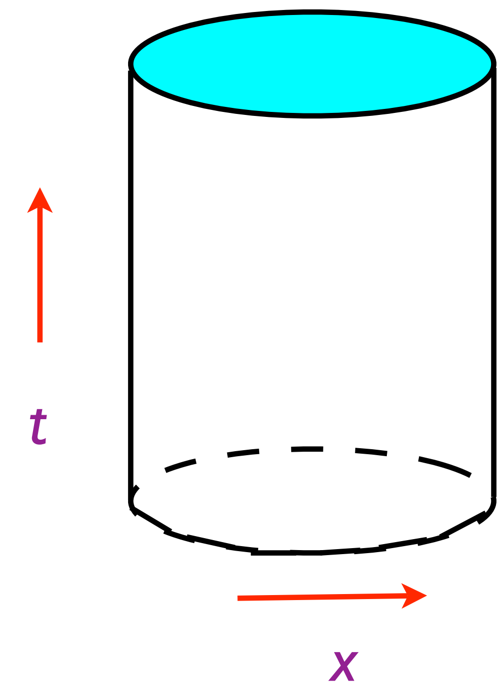
standard Haar measures

standard Gauge action **with SF b.c.'s**

- the integration over Ω ensures that the SF is invariant under gauge transformations of the boundary fields C and C'
- the SF is the quantum mechanical transition amplitude from a state $|C>$ to a state $|C'>$ within time L
- we must extend this formalism to QCD by including fermions

Schrödinger Functional in the continuum

- the SF scheme is defined in a finite L^4 volume, with periodic boundary conditions (b.c.'s) in space and Dirichlet b.c.'s in time
- Dirichlet boundary conditions for quarks imply that we must fix only half of the components of the fermion fields at the boundaries
- with such b.c.'s the (first order) Dirac operator has a unique solution



$$\begin{aligned} P_+\psi\big|_{x_0=0} &= \rho \\ \bar{\psi}P_-|_{x_0=0} &= \bar{\rho} \end{aligned}$$

Dirichlet b.c.'s at $x_0 = 0$

Dirichlet b.c.'s at $x_0 = L$

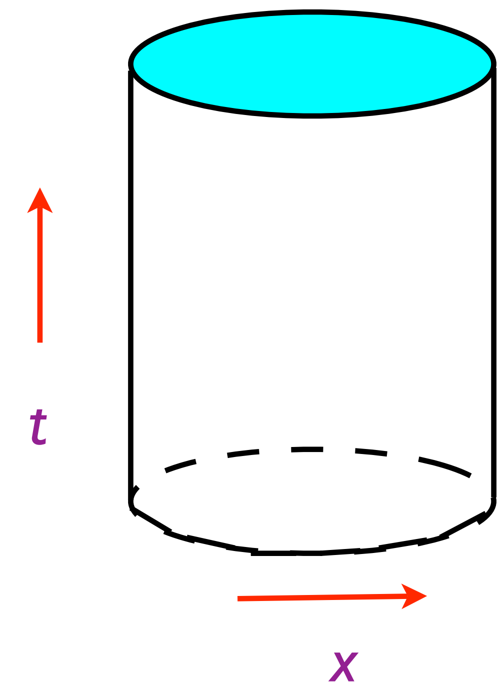
$$\begin{aligned} P_-\psi\big|_{x_0=L} &= \rho' \\ \bar{\psi}P_+|_{x_0=L} &= \bar{\rho}' \end{aligned}$$

$$P_\pm = \frac{1}{2}(1+\gamma_0)$$

projects +ve (-ve) energy field components; i.e. forward (backward) movers

Schrödinger Functional in the continuum

- the SF scheme is defined in a finite L^4 volume, with periodic boundary conditions (b.c.'s) in space and Dirichlet b.c.'s in time
- Dirichlet boundary conditions for quarks imply that we must fix only half of the components of the fermion fields at the boundaries
- with previous b.c.'s the quantum mechanical interpretation of the SF is analogous to that of the Yang Mills theory



$$\mathcal{Z}[C', \bar{\rho}', \rho'; C, \bar{\rho}, \rho] \; = \; \int \mathcal{D}[A]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \exp\{-S[A, \psi, \bar{\psi}]\}$$

$$S[A, \psi, \bar{\psi}] \; = \; S_{\rm QCD}[A, \psi, \bar{\psi}] - \int d^3x [\bar{\psi}(x) P_- \psi(x)]_{x_0=0} - \int d^3x [\bar{\psi}(x) P_- \psi(x)]_{x_0=L}$$

bulk action

d=3 counter-terms due to the SF boundary

- the existence of d ≤ 3 boundary counter-terms is believed to be a general result; there is a lot of corroborative evidence for it
- these counter-terms induce multiplicative renormalization of the boundary fields ρ, ρ', etc.
- thus for vanishing ρ, ρ', etc., the only SF renormalization is that of the mass and the coupling

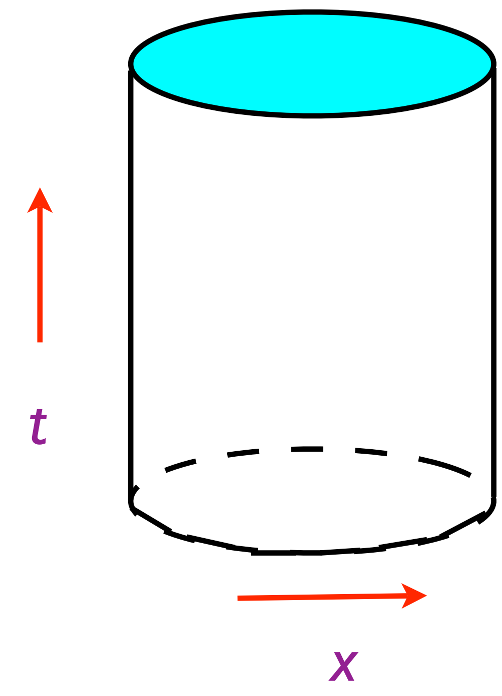
Schrödinger Functional renormalization scheme: gauge coupling

SF scheme: gauge coupling

- the background gauge field configuration B_μ minimizes the action for specific configurations of boundary fields C_k and C_k'

- the effective action is defined as $\Gamma[B] \equiv -\ln \mathcal{Z}[C_k; C_k']$

- its perturbative expansion is



$$\Gamma[B] \equiv -\ln \mathcal{Z}[C';C] = \frac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + g_0^2\Gamma_2[B] + \ldots$$

$$\boxed{\Gamma_0[B] = g_0^2 \, S[B]}$$

- we need to define a coupling which depends only on a single scale; the available one is $1/L$

- it is possible to parametrize C_k and C_k' in terms of a dimensionless parameter η, so that LB depends on η; i.e. the field strength scales as $1/L$

- a choice for the renormalized coupling (i.e. a renormalization scheme) is the definition

$$\bar{g}^2(L) = \left[\frac{\partial\Gamma_0}{\partial\eta} \Big/ \frac{\partial\Gamma}{\partial\eta}\right]_{\eta=0}$$

SF scheme: gauge coupling

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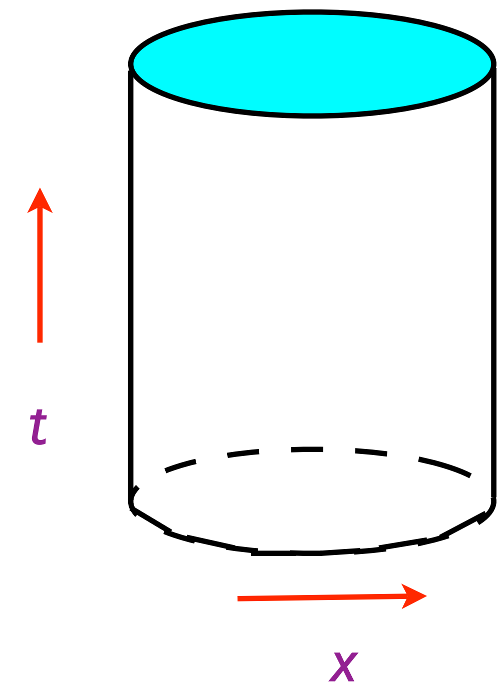

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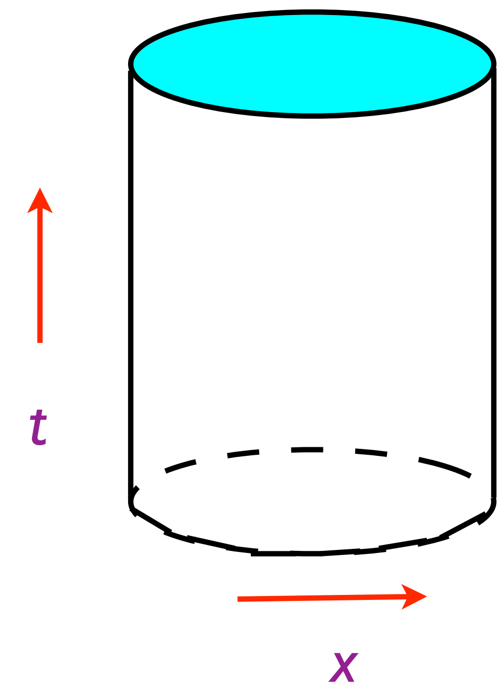
- we need to define a coupling which depends only on a single scale; the available one is $1/L$
- it is possible to parametrize C_k and C_k' in terms of a dimensionless parameter η, so that LB depends on η; i.e. the field strength scales as $1/L$
- other definitions (i.e. other schemes) are possible, e.g.

$$\bar{g}^2(L) \;=\; \left[\frac{3}{4} r^2 F_{q\bar{q}}(r,L)\right]_{r=L/2}$$

force between static quarks at distance r in a box L

SF scheme: gauge coupling

- the background gauge field configuration B_μ minimizes the action for specific configurations of boundary fields C_k and C_k'
- the effective action is defined as $\Gamma[B] \equiv -\ln \mathcal{Z}[C_k; C_k']$
- its perturbative expansion is



$$\Gamma[B] \equiv -\ln \mathcal{Z}[C'; C] = \frac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + g_0^2\Gamma_2[B] + \ldots$$

$$\boxed{\Gamma_0[B] = g_0^2 \; S[B]}$$

- the SF coupling has the following attractive features:
  - depends on a single scale $\mu = 1/L$
  - is an inherently non-perturbative definition
  - the SF b.c.'s exclude gluon zero modes; coupling may be computed even at small boxes L^3
  - relation between S.F. and MS has been worked out in PT

$$\bar{g}^2(L) = \left[\frac{\partial \Gamma_0}{\partial \eta} / \frac{\partial \Gamma}{\partial \eta}\right]_{\eta=0}$$

$$\alpha_{\rm SF}(L) = \alpha_{\overline{\rm MS}}(\mu) + \left[\frac{11}{2\pi}\ln(\mu L) - 1.2556\right]\alpha_{\overline{\rm MS}}(\mu)^2$$

Step scaling function

- we define (in the continuum) a discrete version of the β-function, the step scaling function σ
- it describes the change of the coupling between an (inverse) scale *L* and an an (inverse) scale *sL*, for *s* integer (typically *s*=2)

$$\bar{g}^2(L) \; = \; u \qquad\qquad \bar{g}^2(s\,L) \; = \; u' \qquad\qquad \sigma(s,u) \; = \; u'$$

- this is a discrete form of the Callan-Symanzik beta function

$$\beta(\bar{g}) \; = \; \mu \, \frac{\partial \bar{g}}{\partial \mu}$$

- differentiate both sides w.r.t. μ d/dμ = - L d/dL and use above

$$\beta[\sqrt{\sigma(s,u)}] \; = \; \beta[\sqrt{u}] \, \sqrt{\frac{u}{\sigma(s,u)}} \, \frac{d\sigma(s,u)}{du}$$

- so if we know the ssf, we can reconstruct the Callan-Symanzik function recursively
- the step scaling function in PT is given by

$$\sigma(s,u) \; = \; u + 2b_0 \ln(s)\, u^2 + \cdots$$

Step scaling function

- we next define (in the continuum) a discrete version of the β-function, the step scaling function
- it describes the change of the coupling between an (inverse) scale *L* and an an (inverse) scale *sL*, for *s* integer (typically *s*=2)

$$\bar{g}^2(L) \;=\; u \qquad\qquad \bar{g}^2(s\,L) \;=\; u' \qquad\qquad \sigma(s,u) \;=\; u'$$

- this setup is suitable for a NP computation of the coupling / step scaling function
- in practice we compute NP-ly the step scaling function in a range of couplings u_{min} and u_{max}, corresponding to two scales μ_{max} and μ_{min}; so we obtain the RG-running between them
- the two scales are separated by a power of s, i.e. $\mu_{max} = s^k\, \mu_{min}$, typically s=2
- the gauge coupling and step scaling function calculations requires choosing a regularization: lattice is the obvious choice
- on the lattice it has an additional dependence on the lattice resolution *L/a*

$$\Sigma(s,u,a/L) \;=\; u' \qquad\qquad \sigma(s,u) \;=\; \lim_{a\to 0} \Sigma(s,u,a/L)$$

Step scaling function

- lattice gauge action of choice is the Wilson plaquette one, with some care at the t-boundaries
- lattice fermion action of choice is Wilson, with some care at the t-boundaries
- proceed as follows:

★ choose a lattice with *L/a* points in each direction

★ tune bare coupling so that the renormalized coupling has a fixed value

$$g_0^2 \;\rightarrow\; \bar{g}^2(L) = u$$

★ at the same bare coupling, compute the renormalized coupling on a lattice twice as big *2L/a*

$$\begin{aligned} g_0^2 \quad &\rightarrow \quad \bar{g}^2(2L) \;=\; u' \\ u' \quad &= \quad \Sigma(2,u,a/L) \end{aligned}$$

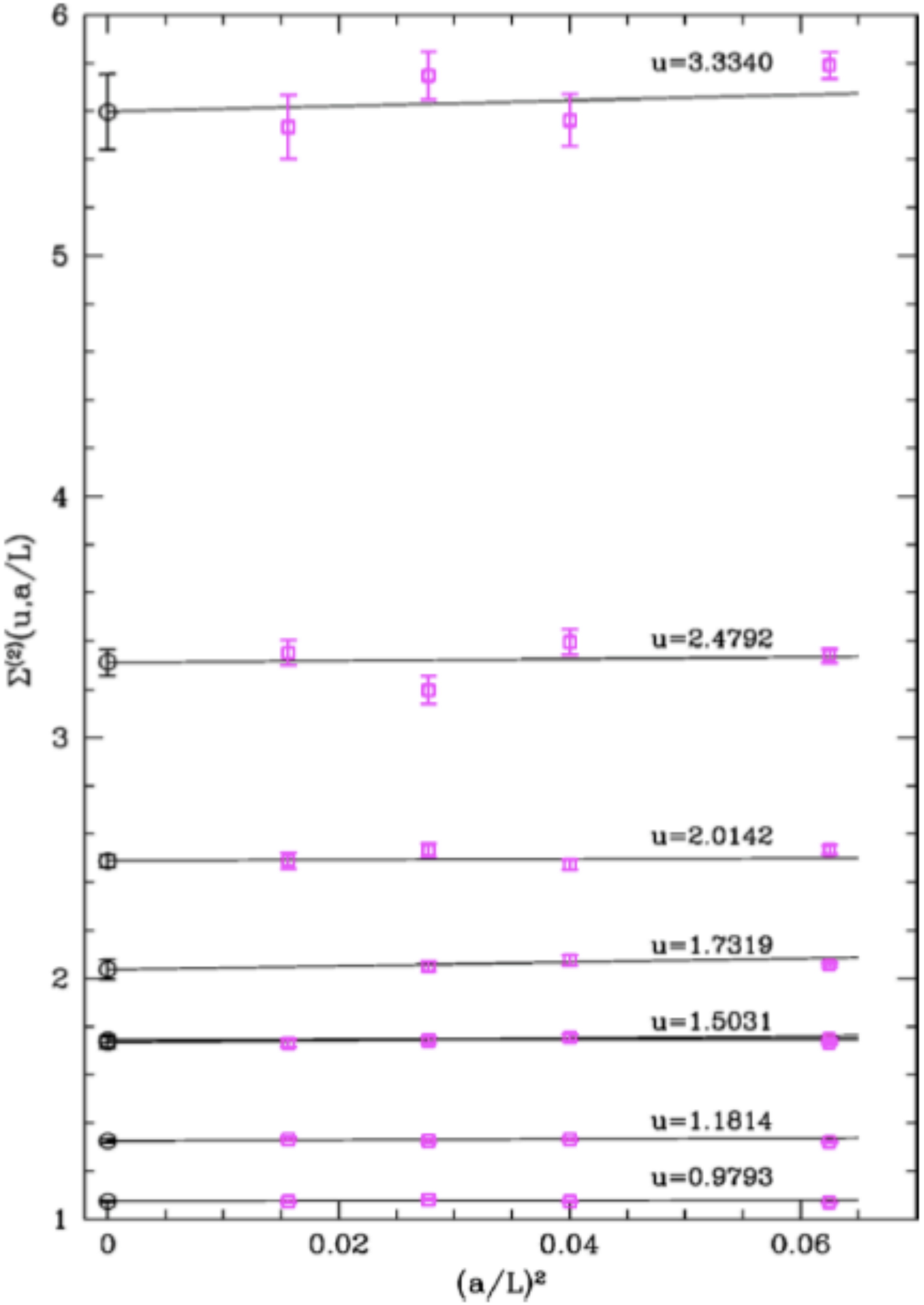
★ repeat this for several resolutions *L'/a, L''/a*

★ extrapolate to the continuum

$$\sigma(s,u) \;=\; \lim_{a\rightarrow 0} \Sigma(s,u,a/L)$$

$$\bar{g}^2(L) \;=\; \left[\frac{\partial\Gamma_0}{\partial\eta}\Big/\frac{\partial\Gamma}{\partial\eta}\right]_{\eta=0}$$

Step scaling function: results for $N_f = 2$



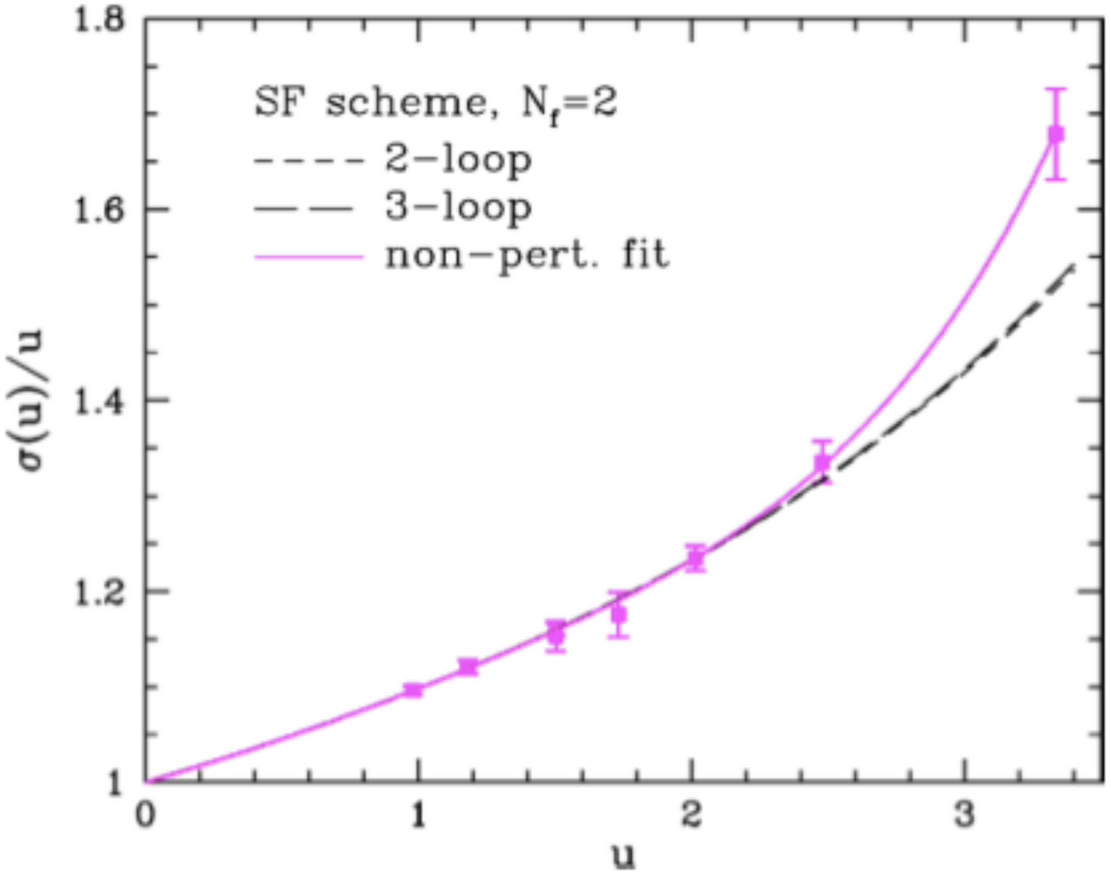
M.Della Morte et al. Nucl.Phys.B713(2005)378

$$g_0^2 \;\to\; \bar{g}^2(L) = u$$

$$\begin{aligned} g_0^2 &\to \bar{g}^2(2L) = u' \\ u' &= \Sigma(2, u, a/L) \end{aligned}$$

$$\sigma(s,u) = \lim_{a\to 0} \Sigma(s,u,a/L)$$

Step scaling function: results for N_f = 2



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$$g_0^2 \rightarrow \bar{g}^2(L) = u$$

$$\begin{aligned} g_0^2 &\rightarrow \bar{g}^2(2L) = u' \\ u' &= \Sigma(2, u, a/L) \end{aligned}$$

$$\sigma(s, u) = \lim_{a \to 0} \Sigma(s, u, a/L)$$

- an expression of the continuum ssf $\sigma(u)$, as a function of the coupling u, is obtained by fitting the points above; so we know the ssf in a range $[u_{min}, u_{max}]$, corresponding to a range of (still unknown!) scales $[\mu_{max}, \mu_{min}]$ (or equivalently $[L_{min}, L_{max}]$)
- NB: the agreement/disagreement between PT/NP is a scheme-dependent observation

Step scaling function: results for $N_f = 2$

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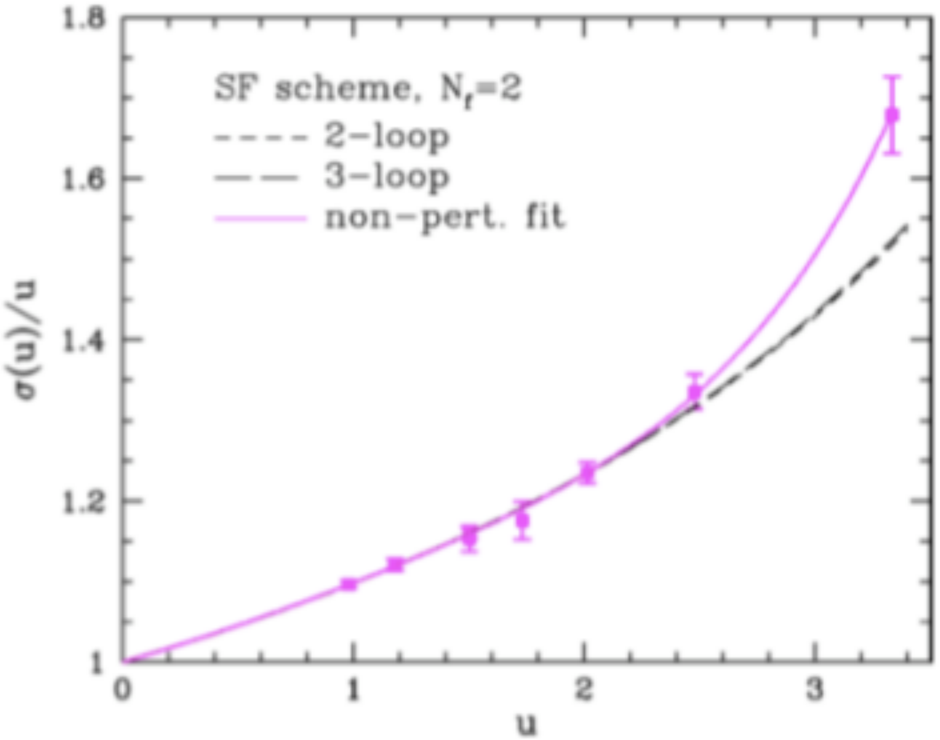

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- NB: the agreement between PT/NP at low couplings is scheme dependent!!

Step scaling function: results for $N_f = 2$

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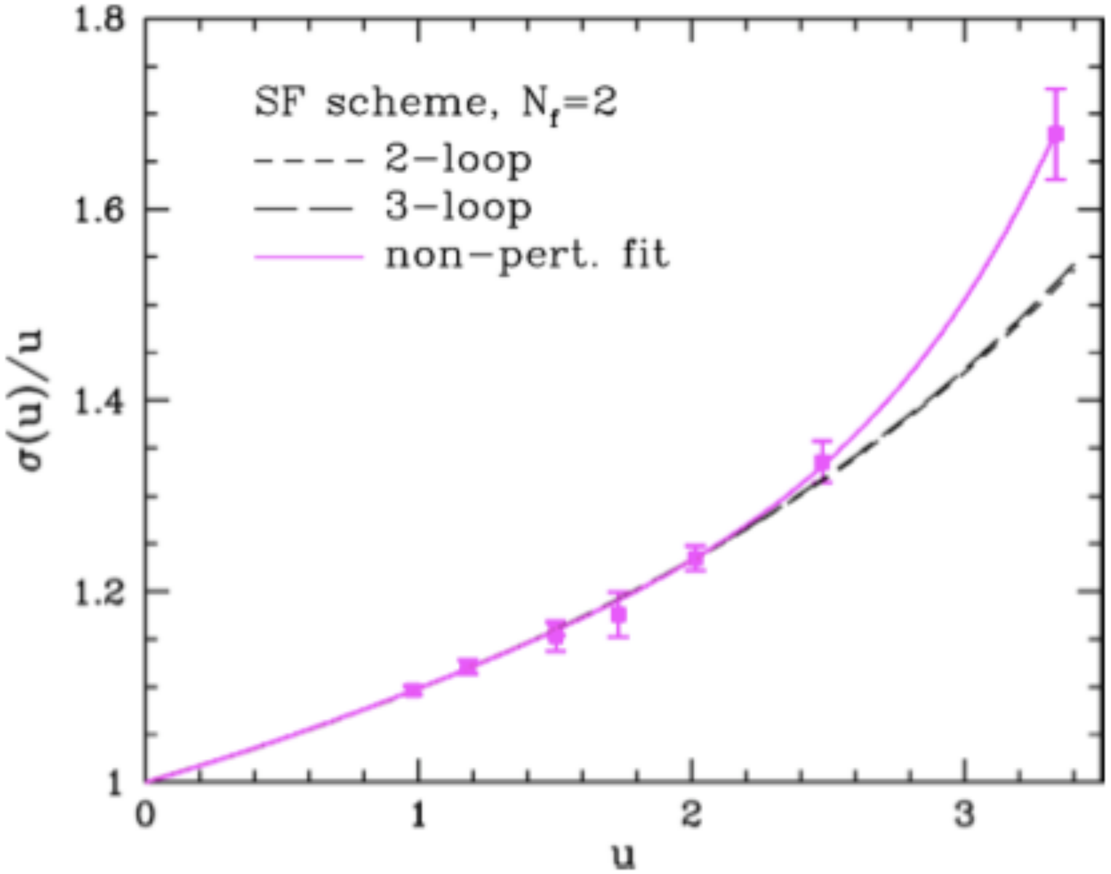

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- NB: the agreement/disagreement between PT/NP is a scheme-dependent observation

Gauge coupling: results for $N_f = 2$

- knowing NPly ssf $\sigma(u)$, we can now compute NP-ly the running strong coupling:
- on the previous plot of $\sigma(u)$ vs. u, choose a number of discrete couplings:

$$
\begin{array}{rcl}
u_1 &=& \bar{g}^2(L_{\min}) \quad \leftrightarrow \quad \dfrac{\Lambda_{\rm SF}}{\mu_{\max}} \\
u_2 &=& \bar{g}^2(2L_{\min}) \quad \leftrightarrow \quad \dfrac{\Lambda_{\rm SF}}{\mu_{\max}/2} = \dfrac{\cancel{\mu_{\max}}}{\mu_{\max}/2}\dfrac{\Lambda_{\rm SF}}{\cancel{\mu_{\max}}} \\
u_3 &=& \bar{g}^2(4L_{\min}) \quad \leftrightarrow \quad \dfrac{\Lambda_{\rm SF}}{\mu_{\max}/4} = \dfrac{\cancel{\mu_{\max}/2}}{\mu_{\max}/4}\dfrac{\cancel{\mu_{\max}}}{\cancel{\mu_{\max}/2}}\dfrac{\Lambda_{\rm SF}}{\cancel{\mu_{\max}}} \\
& \cdots & \quad \cdots \quad \cdots \\
u_k &=& \bar{g}^2(2^k L_{\min} = L_{\max}) \quad \leftrightarrow \quad \dfrac{\Lambda_{\rm SF}}{\mu_{\min}} = \dfrac{\cancel{2\mu_{\min}}}{\mu_{\min}}\dfrac{4\mu_{\min}}{\cancel{2\mu_{\min}}} \cdots \dfrac{\cancel{\mu_{\max}/2}}{\mu_{\max}/4}\dfrac{\cancel{\mu_{\max}}}{\cancel{\mu_{\max}/2}}\dfrac{\Lambda_{\rm SF}}{\cancel{\mu_{\max}}}
\end{array}
$$

known from PT

$$
\frac{\Lambda_{\rm SF}}{\mu_{\max}} = \exp\left[-\frac{1}{2b_0\bar{g}^2(\mu_{\max})}\right]\left[b_0\bar{g}^2(\mu_{\max})\right]^{-b_1/(2b_0^2)}\exp\left[-\int_0^{\bar{g}(\mu_{\max})} dg\left[\frac{1}{\beta(g)}+\frac{1}{b_0 g^3}-\frac{b_1}{b_0^2 g}\right]\right]
$$

$u_l = u_{min}$

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\end{aligned}
$$

known from PT

iteratively work out couplings $u(L)$ and $u(2L)$ for each pair of successive scales μ and $\mu/2$ from ssf $\sigma(u)$

thus we obtain the correspondence between $u(L)$ and $\Lambda_{\mathrm{SF}}/\mu$ (with $\mu = 1/L$) for the whole range of scales μ

Gauge coupling: results for $N_f = 2$

- knowing NPly ssf $\sigma(u)$, we can now compute NP-ly the running strong coupling:
- on the previous plot of $\sigma(u)$ vs. u, choose a number of discrete couplings:

$$u_1 = \bar{g}^2(L_{\min}) \quad \leftrightarrow \quad \frac{\Lambda_{\rm SF}}{\mu_{\max}} \longleftarrow \boxed{\text{known from PT}}$$

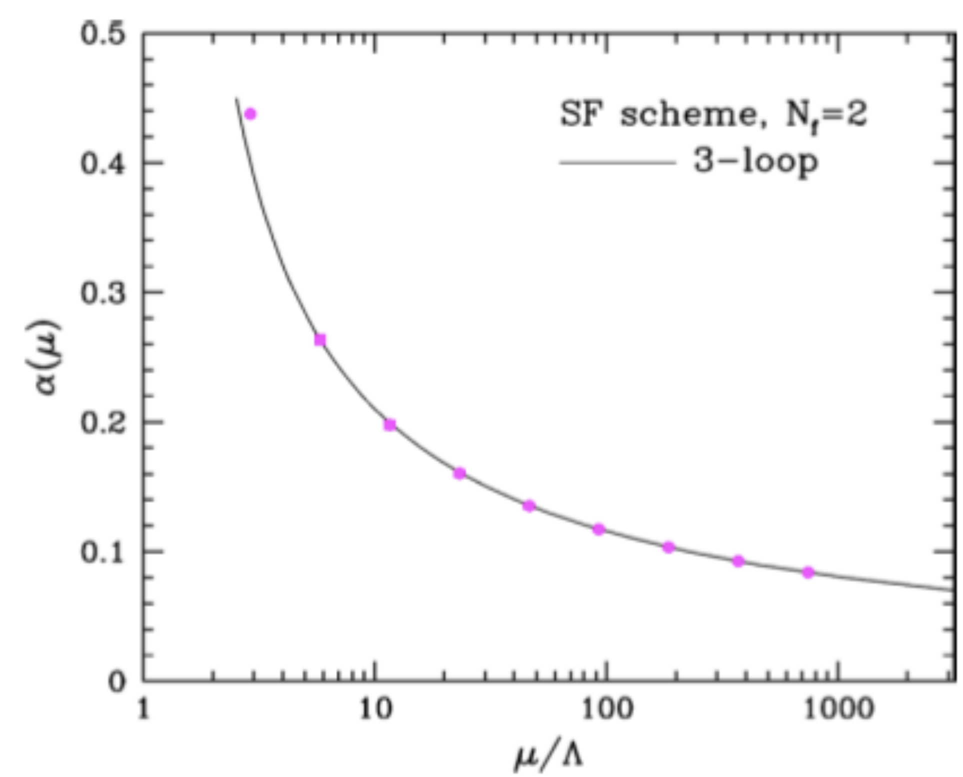
$$u_2 = \bar{g}^2(2L_{\min}) \quad \leftrightarrow \quad \frac{\Lambda_{\rm SF}}{\mu_{\max}/2} = \frac{\cancel{\mu_{\max}}}{\mu_{\max}/2}\,\frac{\Lambda_{\rm SF}}{\cancel{\mu_{\max}}}$$

$$u_3 = \bar{g}^2(4L_{\min}) \quad \leftrightarrow \quad \frac{\Lambda_{\rm SF}}{\mu_{\max}/4} = \frac{\cancel{\mu_{\max}/2}}{\mu_{\max}/4}\,\frac{\cancel{\mu_{\max}}}{\cancel{\mu_{\max}/2}}\,\frac{\Lambda_{\rm SF}}{\cancel{\mu_{\max}}}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$u_k = \bar{g}^2(2^k L_{\min} = L_{\max}) \quad \leftrightarrow \quad \frac{\Lambda_{\rm SF}}{\mu_{\min}} = \frac{\cancel{2\mu_{\min}}}{\mu_{\min}}\,\frac{4\mu_{\min}}{\cancel{2\mu_{\min}}}\cdots\frac{\cancel{\mu_{\max}/2}}{\mu_{\max}/4}\,\frac{\cancel{\mu_{\max}}}{\cancel{\mu_{\max}/2}}\,\frac{\Lambda_{\rm SF}}{\cancel{\mu_{\max}}}$$

M.Della Morte et al. Nucl.Phys.B713(2005)378



- NB: again the scale μ is expressed in units of the (still unknown) $\Lambda_{\rm SF}$; we need to know μ in physical units e.g. GeV

Physical scale

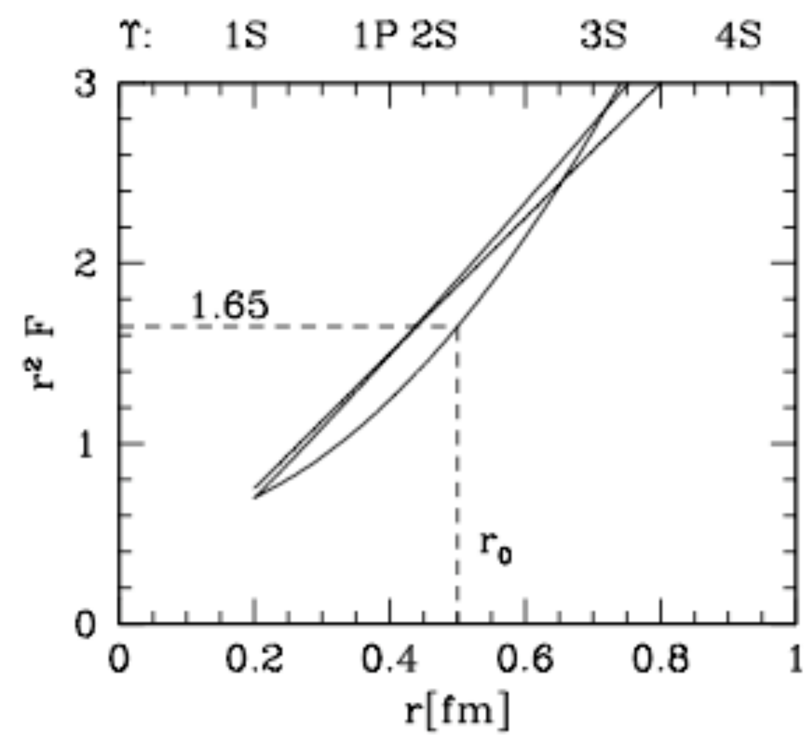
- all results obtained so far are "purely field theoretic"; i.e. they have been obtained from the massless QCD action, without any external (experimental) input
- this is the reason that everything so far involved dimensionless quantities
- in order to make contact with the real world, we need to know μ_{min} (or Λ_{SF}) in physical units
- strategy:
  - for a series of lattice resolutions L_{max}/a, L_{max}/a', L_{max}/a'', ... , tune the bare couplings g_0, g_0', g_0'', ... so as to have the same fixed renormalized coupling $g_R(L_{max}) = const.$
  - for these bare couplings compute some suitable physical quantity; e.g. the proton mass am_p, $a'm_p$, $a''m_p$, ...
  - the products $[L_{max}/a]\times[am_p]$, extrapolated to the continuum for all lattice spacings a, a', a'', ... , gives us $L_{max}\, m_p$
  - use the physical (expt.lly known) value of m_p to get L_{max} (i.e. μ_{min}) and thus Λ_{SF}
- for historical (quenching) and practical reasons, another observable known as the Sommer parameter r_0 is used instead of m_p

Physical scale

- the parameter r_0 is the physical distance at which the static quark-antiquark potential F(r) has a chosen fixed value:

$$\left[r^2 \ F(r)\right]_{r=r_0} \ = \ 1.65$$

dimensionless quantity



- phenomenological models suggest that for [r^2 $F(r)$] = 1.65, we get r_0 = 0.5 fm

- the rest is similar to the procedure described previously, based on the proton mass; instead of m_p, we have $1/r_0$

- so we are in a position to compute μ_{min} (or Λ_{SF}) in physical units

- however, people prefer to see $\Lambda_{\overline{\mathrm{MS}}}$

- this implies that we have to match the SF scheme to $\overline{\mathrm{MS}}$

Λ-dependence of renormalization scheme

- given two schemes "1" and "2", the corresponding renormalized couplings are connected, to all orders in PT by the relation

$$\bar{g}_1^2 \;=\; \bar{g}_2^2 \,\left[1 + c_1\,\bar{g}_2^2 + c_2\,\bar{g}_2^4 + c_3\,\bar{g}_2^6 \cdots\right]$$

- recall that the corresponding Λ parameters are written as

$$\Lambda_{1,2} \;=\; \lim_{\mu_0 \to \infty} \mu_0 \;\exp\left[-\frac{1}{2b_0\bar{g}_{1,2}^2(\mu_0)}\right]\,\left[b_0\bar{g}_{1,2}^2(\mu_0)\right]^{-b_1/(2b_0^2)}$$

- from these expressions we can work out the ratio, valid to all orders in PT

$$\frac{\Lambda_1}{\Lambda_2} \;=\; \exp\left[\frac{c_1}{2b_0}\right]$$

- NB: only the first perturbative coefficient is necessary!!
- the scheme matching has been worked out between SF and $\overline{\mathrm{MS}}$

M.Della Morte et al. Nucl.Phys.B713(2005)378

$$\Lambda_{\overline{\mathrm{MS}}}^{N_f=2} \;=\; 245(16)(16)\mathrm{MeV} \qquad \text{with } r_0 = 0.5\mathrm{fm}$$

Schrödinger Functional renormalization scheme: quark mass

Quark mass RG-running and the SF

- having dealt with the gauge coupling we turn to the other QCD fundamental parameters, i.e. the quark masses
- they are "unphysical" (i.e. non-observable) field theoretic quantities, which depend on the renormalization scale
- their RG-running is governed by the **anomalous dimension** γ
- in a mass independent scheme, $\gamma(g_R)$ depends on the number of flavours but not on the quark masses
- it is defined as: $$m_R\,\gamma(g_R) = \mu\,\frac{\partial m_R}{\partial \mu}$$
- and has the following perturbative expansion:

$$\gamma(g) = -g^2\left[d_0 + d_1\,g^2 + d_2\,g^4 + \cdots\right]$$

$d_0 = \dfrac{8}{(4\pi)^2}$ universal

renormalization scheme dependent

Quark mass RG-running and the SF

- the quark mass RG equation is integrated between a minimum and a maximal energy scale; the former is taken to infinity (i.e. coupling to zero)
- this procedure is similar to that exposed in detail for the gauge coupling, and gives rise to a constant quantity, with the dimensions of mass

$$M_{\rm RGI} \equiv \lim_{\mu_0 \to \infty} m_{\rm R}(\mu_0) \left[2b_0 \; g^2_{\rm R}(\mu_0) \right]^{-d_0/(2b_0)}$$

$$M_{\rm RGI} = m_{\rm R}(\mu) \left[2b_0 \; g^2_{\rm R}(\mu) \right]^{-d_0/(2b_0)} \exp \left[- \int_0^{g_{\rm R}(\mu)} dg \; \left[\frac{\gamma(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right]$$

regular in the limit $g_R(\mu_0) \to 0$

- the ratio of the RGI mass $M_{\rm RGI}$ to the renormalized mass $m_{\rm R}(\mu)$ is a field theoretic quantity, independent of any physical input
- it depends on the flavour number, but not on the quark mass value
- using the definition of the RGI mass for two distinct schemes, it can be shown that it is a scheme independent quantity

Quark mass RG-running and the SF

- the definition of the quark mass step scaling function is the ratio of the renormalized masses at two consecutive scales, at

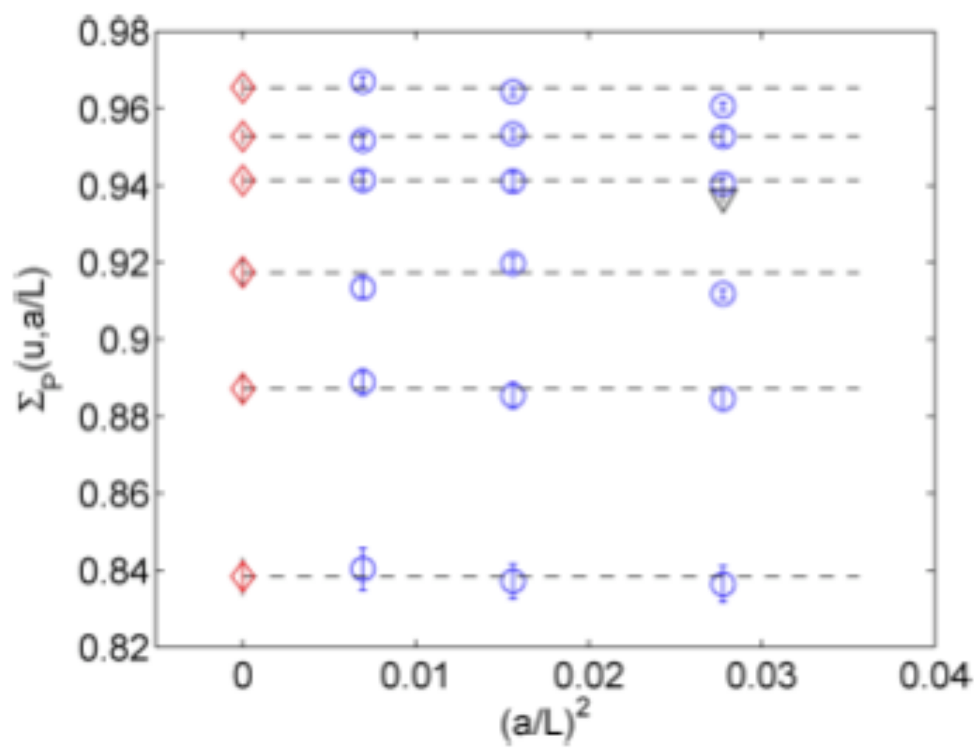
$$\sigma_P(s,u) = \frac{m_{\rm R}(\mu)}{m_{\rm R}(\mu/s)} = \frac{Z_P^{-1}(aL)\,\cancel{m_0}(g_0)}{Z_P^{-1}(asL)\,\cancel{m_0}(g_0)} = \frac{Z_P^{-1}(aL)}{Z_P^{-1}(asL)}$$

at same renorm. coupling u

$g_0{}^2$ corresponding to u

this is how it is computed

- computation performed at zero quark mass (i.e. ssf defined in the chiral limit)
- it follows recursive logic of the coupling ssf computation
- the lattice ssf $\Sigma_P(u,L)$ is computed at several renormalized couplings and extrapolated to the continuum limit
- the $N_f = 2$ result is shown



M.Della Morte et al. Nucl.Phys.B729(2005)117

Quark mass RG-running and the SF

- knowing NPly ssf $\sigma_P(u)$, we can now compute NP-ly the running strong coupling

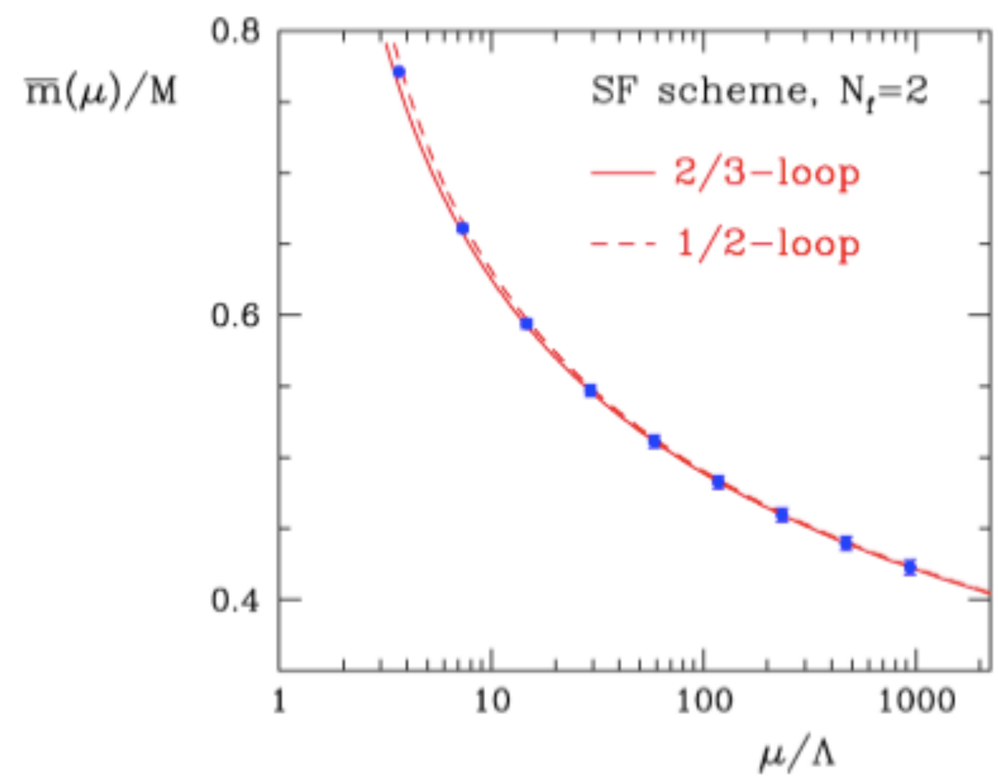
$$\frac{M}{m_{\rm R}(\mu_{\rm min})} = \frac{\cancel{m_{\rm R}(2\mu_{\rm min})}}{m_{\rm R}(\mu_{\rm min})}\,\frac{\cancel{m_{\rm R}(4\mu_{\rm min})}}{\cancel{m_{\rm R}(2\mu_{\rm min})}}\cdots\frac{\cancel{m_{\rm R}(\mu_{\rm max}/2)}}{\cancel{m_{\rm R}(\mu_{\rm max}/4)}}\,\frac{\cancel{m_{\rm R}(\mu_{\rm max})}}{\cancel{m_{\rm R}(\mu_{\rm max}/2)}}\,\frac{M}{\cancel{m_{\rm R}(\mu_{\rm max})}}$$

known from ssf $\sigma_P(u)$

$$\frac{M}{m_{\rm R}(\mu_{\rm max})} = \left[2b_0\bar{g}^2(\mu_{\rm max})\right]^{-d_0/(2b_0^2)} \exp\left[-\int_0^{\bar{g}(\mu_{\rm max})} dg\,\left[\frac{\gamma(g)}{\beta(g)} - \frac{d_0}{b_0 g}\right]\right]$$

u_{min}

$$\frac{M}{m_{\rm R}(\mu_{\rm min})} = 1.297(16) \qquad N_f = 2$$



M.Della Morte et al. Nucl.Phys.B729(2005)117

Quark mass RG-running and the SF

- now the RGI quark mass of a given flavour f can be computed

$$M_f \;=\; \frac{M_f}{m_{\mathrm{R}}(\mu_{\min})}\; m_{\mathrm{R}}(\mu_{\min}) \;=\; \frac{M_f}{m_{\mathrm{R}}(\mu_{\min})}\; \lim_{a\to 0}\; Z_P^{-1}(a\mu_{\min}, g_0)\; m_{\mathrm{PCAC}}(g_0)$$

scheme independent

scheme dependent
scale dependent

known in the C.L.

regularization
dependent
scale dependent

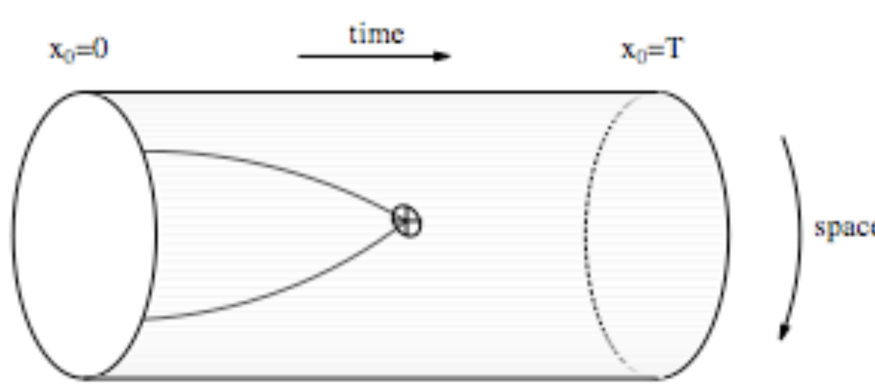
must be
computed NPly

regularization
dependent
scale dependent

flavour
dependence

- the bare PCAC quark mass is defined as

$$m_{\mathrm{PCAC}} \;=\; \frac{Z_A\;\partial_0\; f_A}{2\; f_P}$$



- for SF correlation functions:

$$\begin{aligned} f_P &= \langle\; P(x)\; O(0)\;\rangle \\ f_A &= \langle\; A_0(x)\; O(0)\;\rangle \end{aligned}$$

boundary source composite field with
pseudoscalar quantum numbers

Quark mass RG-running and the SF

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$$M_f = \frac{M_f}{m_{\rm R}(\mu_{\rm min})}\, m_{\rm R}(\mu_{\rm min}) = \frac{M_f}{m_{\rm R}(\mu_{\rm min})} \lim_{a\to 0} Z_P^{-1}(a\mu_{\rm min}, g_0)\, m_{\rm PCAC}(g_0)$$

scheme independent

scheme dependent
scale dependent

known in the C.L.

regularization
dependent
scale dependent

must be
computed NPly

regularization
dependent
scale dependent

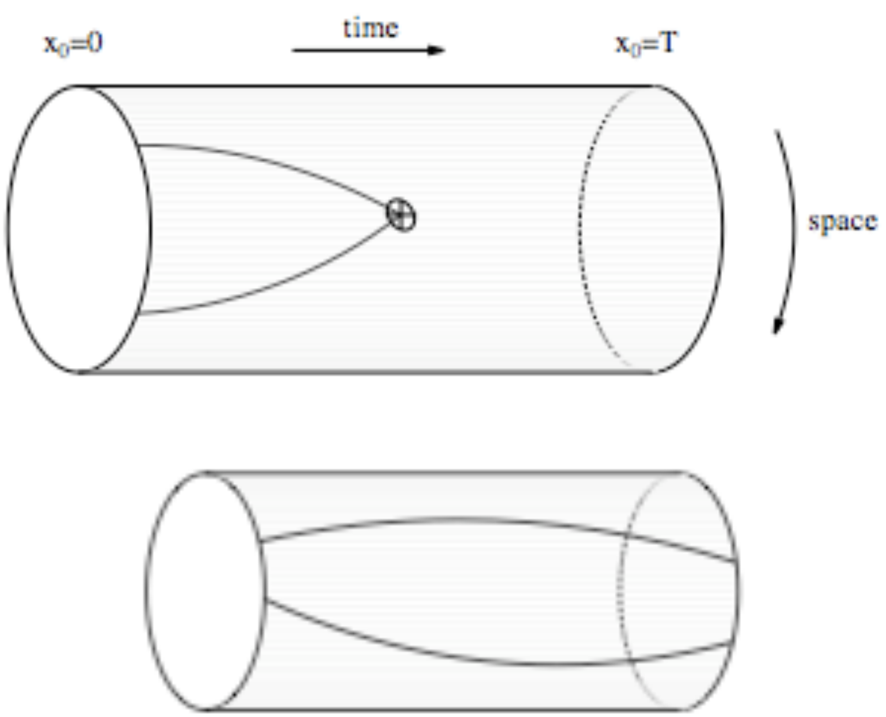
flavour
dependence

- the SF renormalization condition for the pseudoscalar density is:

$$\frac{Z_P(L_{\rm max})\; f_P(L_{\rm max}/2)}{\sqrt{f_1}} = {\rm T.L.}\Big[\frac{f_P(L_{\rm max}/2)}{\sqrt{f_1}}\Big]$$

cancels boundary
quark field
renormalization

tree level



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scheme dependent
scale dependent

must be
computed NPly

flavour
dependence

- simulations at the physical up/down quark masses are a daunting task
- simulations in the mass range [$m_s/4$, m_c] are nowadays feasible
- a nice approach is to define a reference quark mass (approximately $m_s/2$) for which a "Kaon" consisting of two degenerate valence quarks weighs 495 MeV (the "physical" value)
- this "world" is a two degenerate flavour (Nf = 2) theory
- the previous SF procedure, once the bare quark mass is tuned to the reference quark mass etc., gives $M_{\rm ref}$ = 72 (3) (13) MeV
- next use the chiral PT result $M_s = 48/25\, M_{\rm ref}$, to obtain $M_{\rm strange}$ = 138 (5) (26) MeV

M.Della Morte et al. Nucl.Phys.B729(2005)117

Recapitulation of RG-running with the SF

PT-regime: compute RGI quantities (Λ_{QCD}, M_{RGI}); use them in high-energy phenomena (jet Physics)

$$\mu = 2^n / L_{\rm max}$$

renormalization scale (energy)

n recursive steps

hadronic scheme

$$\mu = 1/L_{\rm max}$$

$$L_{\rm max} \sim 0.5 {\rm fm}$$

Schrödinger Functional scheme

$$g_{\rm R} \quad \leftrightarrow \quad m_p^{\rm phys}$$

$$m_{\rm R}^{u/d} \quad \leftrightarrow \quad m_\pi^{\rm phys}$$

NP-regime: compute hadronic matrix elements and SF renorm. constants