Lattice QCD and Non-Perturbative Renormalization

GDR-Workshop

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Lecture 2: Schrödinger Functional
RG-running on the lattice: motivation
Operator RG-running

• suppose a quantity $Q(\mu)$ (quark mass, operator WME) is renormalized in a NP scheme

$$Q_R(\mu) = \lim_{a \to 0} Z_Q(g_0^2, a \mu) Q(g_0^2)$$

• if you use a hadronic scheme, the renormalization scale is going to be low $\mu \sim m_H$

• you need to know $Q(\mu)$ at a larger scale either for conventional reasons (e.g. people are used to MS-scheme quark masses $m_q(\mu)$ with $\mu \sim 2\text{GeV}$) or for matching with perturbative scales, as in the OPE:

$$Q^{\text{phys}} = \sum C_W(\mu) \lim_{a \to 0} \left[ Z_Q(g_0^2, a \mu) < f \left| Q(g_0^2) \right| i > \right]$$
Operator RG-running

• suppose a quantity $Q(\mu)$ (quark mass, operator WME) is renormalized in a NP scheme

$$Q_R(\mu) = \lim_{a \to 0} Z_Q(g_0^2, a\mu) \cdot Q(g_0^2)$$

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---

Wilson coefficients calculated in PT short-distance effects

renormalization scale must be large; say 10GeV
Operator RG-running

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- If you use a hadronic scheme, the renormalization scale is going to be low $\mu \sim m_H$

- You need to know $Q(\mu)$ at a larger scale either for conventional reasons (i.e., people are used to MS-scheme quark masses $m_q(\mu)$ with $\mu \sim 2\text{GeV}$) or for matching with perturbative scales, as in the OPE:

\[ Q_{\text{phys}} = \sum C_W(\mu) \lim_{a \to 0} \left[ Z_Q(g_0^2, a \mu) \, \langle f \, | \, Q(g_0^2) \, | \, i \rangle \right] \]

must be $O(1)$, so as to avoid large logs

must be smaller than 1, so as to avoid discretization errors
Operator RG-running

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• if we wish to compute everything at one go (a single lattice) we must also ensure that $m_H L >> 1$, in order to avoid finite size errors

• i.e. we must satisfy $L >> 1/m_H \sim 1/(0.15 \text{ GeV}) >> 1/\mu \sim 1/(10 \text{ GeV}) > a$

• IMPOSSIBLE on present day resources as it gives $L/a = O(100-1000)$
Operator RG-running

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$$Q^{\text{phys}} = \sum C_W(\mu) \lim_{a \to 0} \left[ Z_Q(g_0^2, a\mu) \, < f \mid Q(g_0^2) \mid i > \right]$$

• need to compute the renormalized WME at a hadronic (low) scale $\mu_{\text{min}}$ and then do RG-running all the way to a perturbative (high) scale $\mu_{\text{max}}$

• an option is using PT for the RG running, introducing ill-controlled $O(g^n)$ systematic errors

• the SF scheme, combined with finite size techniques, is the only one used so far for non-perturbative RG-running
RG-running: generalities
• the basic idea is always that of Callan-Symanzik

• there are mass-independent renormalization schemes, in which the renormalization conditions are imposed at the chiral limit (this is sufficient to remove UV divergences)

• in such schemes the renormalization constants and running functions do not depend on the theory’s masses: $Z_g(a\mu, g_0)$, $Z_m(a\mu, g_0)$, $\beta(g_R)$, $\gamma(g_R)$ etc.

• first we reformulate what we know from continuum QCD renormalization (usually worked out in PT) in a general, non-perturbative (N.P.) language, suitable to N.P. computations

• we start with the RG-running of the gauge coupling, expressed in terms of the Callan-Symanzik $\beta$-function

$$\beta(g_R) = \mu \frac{\partial g_R}{\partial \mu}$$

• it is simple to integrate this from a reference scale $\mu_0$ to a general scale $\mu$

$$\frac{\mu_0}{\mu} = \exp \left[ - \int_{g_R(\mu_0)}^{g_R(\mu)} \frac{dg}{\beta(g)} \right]$$
RG-running in the continuum

\[ \frac{\mu_0}{\mu} = \exp \left[ - \int_{g_R(\mu_0)}^{g_R(\mu)} \frac{dg}{\beta(g)} \right] \]

- it is natural, for an asymptotically free theory (QCD), to choose the reference scale \( \mu_0 \to \infty \), for which \( g_R(\mu_0) \to 0 \)

- we know, however, the perturbative behaviour of the beta function at small couplings

\[ \beta(g) = -g^3 \left[ b_0 + b_1 g^2 + b_2 g^4 + \cdots \right] \]

\[ b_0 = \frac{1}{(4\pi)^2} \left[ 11 - \frac{2N_f}{3} \right] \]

universal

\[ b_1 = \frac{1}{(4\pi)^4} \left[ 102 - \frac{38N_f}{3} \right] \]

renormalization scheme dependent

- the perturbative expression for \( \beta(g_R) \) tells us that the above integral diverges at the lower end \( g_R(\mu_0) = 0 \), due to the first two terms of the expansion (NLO)
RG-running in the continuum

\[ \frac{\mu_0}{\mu} = \exp \left[ - \int \frac{g_R(\mu)}{g_R(\mu_0)} \frac{dg}{\beta(g)} \right] \]

- trick: add and subtract the potentially diverging term \(1/\beta_{\text{NLO}}(g_R)\) in the integrand:

\[ \mu_0 = \mu \exp \left[ - \int \frac{g_R(\mu)}{g_R(\mu_0)} dg \left[ \frac{1}{\beta(g)} - \frac{1}{\beta_{\text{NLO}}(g)} \right] \right] \exp \left[ - \int \frac{g_R(\mu)}{g_R(\mu_0)} \frac{1}{\beta_{\text{NLO}}(g)} \right] \]

regular in the limit \(g_R(\mu_0) \to 0\)

divergent in the limit \(g_R(\mu_0) \to 0\); calculable for \(g_R(\mu_0) \neq 0\)

- calculate the NLO integral (for \(g_R(\mu_0) \neq 0\)) and carry everything that depends on \(\mu_0\) to the LHS, leaving all \(\mu\)-dependent quantities on the RHS

\[ \mu_0 \exp \left[ - \frac{1}{2b_0g_R^2(\mu_0)} \right] \left[ b_0 g_R^2(\mu_0) \right]^{-b_1/(2b_0^2)} = \mu \exp \left[ - \frac{1}{2b_0g_R^2(\mu)} \right] \left[ b_0 g_R^2(\mu) \right]^{-b_1/(2b_0^2)} \exp \left[ - \int \frac{g_R(\mu)}{g_R(\mu_0)} \frac{dg}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g^2} \right] \]
RG-running in the continuum

\[
\mu_0 \exp \left[ - \frac{1}{2b_0 g_R^2(\mu_0)} \right] \left[ b_0 g_R^2(\mu_0) \right]^{-b_1/(2b_0^2)} = \\
\mu \exp \left[ - \frac{1}{2b_0 g_R^2(\mu)} \right] \left[ b_0 g_R^2(\mu) \right]^{-b_1/(2b_0^2)} \exp \left[ - \int_{g_R(\mu_0)}^{g_R(\mu)} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right]
\]

- in the limit \( g_R(\mu_0) \to 0 \), the RHS is \( \mu_0 \) independent; therefore the same holds for the LHS

- this enables us to define an energy scale, typical of the theory

\[
\Lambda_{\text{QCD}} \equiv \lim_{\mu_0 \to \infty} \mu_0 \exp \left[ - \frac{1}{2b_0 g_R^2(\mu_0)} \right] \left[ b_0 g_R^2(\mu_0) \right]^{-b_1/(2b_0^2)}
\]

\[
\Lambda_{\text{QCD}} = \mu \exp \left[ - \frac{1}{2b_0 g_R^2(\mu)} \right] \left[ b_0 g_R^2(\mu) \right]^{-b_1/(2b_0^2)} \exp \left[ - \int_{0}^{g_R(\mu)} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right]
\]
\[ \Lambda_{\text{QCD}} \equiv \lim_{\mu_0 \to \infty} \mu_0 \exp \left[ - \frac{1}{2b_0 g_R^2(\mu_0)} \right] \left[ b_0 g_R^2(\mu_0) \right]^{b_1/(2b_0^2)} \]

\[ \Lambda_{\text{QCD}} = \mu \exp \left[ - \frac{1}{2b_0 g_R^2(\mu)} \right] \left[ b_0 g_R^2(\mu) \right]^{b_1/(2b_0^2)} \exp \left[ - \int_0^{g_R(\mu)} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right] \]

- this is an exact expression, from which standard PT results for LO and NLO cases may be obtained
- the “miracle” of renormalization is that, even for massless QCD, it generates an energy scale
- \( \Lambda_{\text{QCD}} \) is Renormalization Group Invariant (RGI; i.e. \( \mu \)-independent) but depends on the renormalization scheme (\( \beta \) is scheme independent only to NLO order)
- \( \Lambda_{\text{QCD}} \) depends on the number of quark flavours (cf. \( b_0, b_1 \)) but not on the value of the quark masses; in fact it may be calculated in PT, or computed NP with \( N_f \) massless quarks
- already at LO you can see from above that \( \Lambda_{\text{QCD}} \) corresponds to a NP coupling (oxymoron!)

\[ g_R^2(\mu) = - \frac{1}{2b_0 \ln(\mu/\Lambda_{\text{QCD}})} \]
RG-running in the continuum

\[ \Lambda_{\text{QCD}} = \mu \exp \left[ -\frac{1}{2b_0g_R^2(\mu)} \right] [b_0g_R^2(\mu)]^{-b_1/(2b_0^2)} \exp \left[ -\int_0^{g_R(\mu)} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0g^3} - \frac{b_1}{b_0^2g} \right] \right] \]

- suppose we have chosen a scheme; i.e. we have a definition of \( g_R(\mu) \), accompanied by a renormalization condition for the coupling

- suppose that we have developed a powerful NP method (lattice) with which to compute \( \beta(\mu) \) in a vast range of scales: from, say \( \mu_{\text{min}} \sim \Lambda_{\text{QCD}} \) to \( \mu_{\text{max}} \sim 100 \text{ GeV} \)

- the above tells us that the dimensionless ratio \( \Lambda_{\text{QCD}} / \mu \) can be calculated from first principles of QCD, without any “physical” input (e.g. a hadronic mass or any other experimentally known quantity); this ratio is a “pure” Quantum Field Theory quantity

- a “physical” input is required (as shown below) in order to establish the correspondence of a given reference coupling \( g_R(\mu_{\text{ref}}) \) to its scale \( \mu_{\text{ref}} \) (in GeV); from this, \( \Lambda_{\text{QCD}} \) (in GeV) is immediately obtained

- we will show that the Schrödinger Functional renormalization scheme beautifully fulfills these expectations
The Schrödinger Functional

the SF scheme is defined in a finite $L^4$ volume, with periodic boundary conditions (b.c.'s) in space and Dirichlet b.c.'s in time

For a Yang-Mills theory this means that we must specify the gauge configurations at the time boundaries

$$A^\Omega_\mu(x) = \Omega(x)A_\mu(x)\Omega^{-1}(x) + \Omega(x)\partial_\mu\Omega(x)^{-1}$$

Gauge field

Dirichlet b.c.'s at time boundaries

$$A_k(x) = C^\Omega_k(\vec{x}) \quad @ \quad x_0 = 0$$

$$A_k(x) = C'_k(\vec{x}) \quad @ \quad x_0 = L$$

periodic b.c.'s in space

$$A_k(x) = A_k(x + L\vec{k})$$

$$\Omega(\vec{x}) = \Omega(\vec{x} + L\vec{k}) \quad @ \quad x = (\vec{x}, 0)$$
Schrödinger Functional in the continuum

- The SF scheme is defined in a finite $L^4$ volume, with periodic boundary conditions (b.c.’s) in space and Dirichlet b.c.’s in time.

- The Euclidean partition function defines the SF:

$$Z[C', C] = \int D[\Omega] \int D[A_\mu] \exp\{-S_G[A]\}$$

- The integration over $\Omega$ ensures that the SF is invariant under gauge transformations of the boundary fields $C$ and $C'$.

- The SF is the quantum mechanical transition amplitude from a state $|C\rangle$ to a state $|C'\rangle$ within time $L$.

- We must extend this formalism to QCD by including fermions.
Schrödinger Functional in the continuum

- the SF scheme is defined in a finite $L^4$ volume, with periodic boundary conditions (b.c.'s) in space and Dirichlet b.c.'s in time
- Dirichlet boundary conditions for quarks imply that we must fix only half of the components of the fermion fields at the boundaries
- with such b.c.'s the (first order) Dirac operator has a unique solution

\[
\begin{align*}
P^+ \psi \bigg|_{x_0=0} &= \rho \\
\bar{\psi} P^- \bigg|_{x_0=0} &= \bar{\rho}
\end{align*}
\]

Dirichlet b.c.'s at $x_0 = 0$

\[
\begin{align*}
P^- \psi \bigg|_{x_0=L} &= \rho' \\
\bar{\psi} P^+ \bigg|_{x_0=L} &= \bar{\rho}'
\end{align*}
\]

Dirichlet b.c.'s at $x_0 = L$

\[P_{\pm} = \frac{1}{2} (1 + \gamma_0)\]

projects +ve (-ve) energy field components; i.e. forward (backward) movers
Schrödinger Functional in the continuum

- the SF scheme is defined in a finite $L^4$ volume, with periodic boundary conditions (b.c.'s) in space and Dirichlet b.c.'s in time
- Dirichlet boundary conditions for quarks imply that we must fix only half of the components of the fermion fields at the boundaries
- with previous b.c.'s the quantum mechanical interpretation of the SF is analogous to that of the Yang Mills theory

$$ Z[C', \bar{\rho}', \rho'; C, \bar{\rho}, \rho] = \int D[A] D[\psi] D[\bar{\psi}] \exp\{-S[A, \psi, \bar{\psi}]\} $$

$$ S[A, \psi, \bar{\psi}] = S_{QCD}[A, \psi, \bar{\psi}] - \int d^3 x [\bar{\psi}(x)P_\psi(x)]_{x_0=0} - \int d^3 x [\bar{\psi}(x)P_-\psi(x)]_{x_0=L} $$

- the existence of $d \leq 3$ boundary counter-terms is believed to be a general result; there is a lot of corroborative evidence for it
- these counter-terms induce multiplicative renormalization of the boundary fields $\rho, \rho'$, etc.
- thus for vanishing $\rho, \rho'$, etc., the only SF renormalization is that of the mass and the coupling
Schrödinger Functional renormalization scheme: gauge coupling
SF scheme: gauge coupling

- The background gauge field configuration $B_\mu$ minimizes the action for specific configurations of boundary fields $C_k$ and $C_k'$
- The effective action is defined as $\Gamma[B] \equiv -\ln Z[C_k; C_k']$
- Its perturbative expansion is

\[
\Gamma[B] \equiv -\ln Z[C'; C] = \frac{1}{g_0^2} \Gamma_0[B] + \Gamma_1[B] + g_0^2 \Gamma_2[B] + \ldots
\]

\[
\Gamma_0[B] = g_0^2 S[B]
\]

- We need to define a coupling which depends only on a single scale; the available one is $1/L$
- It is possible to parametrize $C_k$ and $C_k'$ in terms of a dimensionless parameter $\eta$, so that $LB$ depends on $\eta$; i.e. the field strength scales as $1/L$
- A choice for the renormalized coupling (i.e. a renormalization scheme) is the definition

\[
g^2(L) = \left[ \frac{\partial \Gamma_0}{\partial \eta} / \frac{\partial \Gamma}{\partial \eta} \right]_{\eta=0}
\]
SF scheme: gauge coupling

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- the effective action is defined as $\Gamma[B] \equiv -\ln Z \left[ C_k; C_k' \right]$
- its perturbative expansion is

$$\Gamma[B] \equiv -\ln Z[C'; C] = \frac{1}{g_0^2} \Gamma_0[B] + \Gamma_1[B] + g_0^2 \Gamma_2[B] + \ldots$$

- $\Gamma_0[B] = g_0^2 S[B]$

- we need to define a coupling which depends only on a single scale; the available one is $1/L$
- it is possible to parametrize $C_k$ and $C_k'$ in terms of a dimensionless parameter $\eta$, so that $LB$ depends on $\eta$; i.e. the field strength scales as $1/L$
- other definitions (i.e. other schemes) are possible, e.g.

$$\bar{g}^2(L) = \left[ \frac{3}{4} r^2 F_{q\bar{q}}(r, L) \right]_{r=L/2}$$

force between static quarks at distance $r$ in a box $L$
SF scheme: gauge coupling

- the background gauge field configuration $B_\mu$ minimizes the action for specific configurations of boundary fields $C_k$ and $C_k'$
- the effective action is defined as $\Gamma[B] \equiv -\ln Z[C_k; C_k']$
- its perturbative expansion is

$$\Gamma[B] \equiv -\ln Z[C'; C] = \frac{1}{g^2_0} \Gamma_0[B] + \Gamma_1[B] + g^2_0 \Gamma_2[B] + \ldots$$

$$\Gamma_0[B] = g^2_0 S[B]$$

- the SF coupling has the following attractive features:
  - depends on a single scale $\mu = 1/L$
  - is an inherently non-perturbative definition
  - the SF b.c.'s exclude gluon zero modes; coupling may be computed even at small boxes $L^3$
  - relation between S.F. and MS has been worked out in PT

$$\bar{g}^2(L) = \left[ \frac{\partial \Gamma_0}{\partial \eta} / \frac{\partial \Gamma}{\partial \eta} \right]_{\eta=0}$$

$$\alpha_{\text{SF}}(L) = \alpha_{\text{MS}}(\mu) + \left[ \frac{11}{2\pi} \ln(\mu L) - 1.2556 \right] \alpha_{\text{MS}}(\mu)^2$$
Step scaling function

- we define (in the continuum) a discrete version of the $\beta$-function, the step scaling function $\sigma$
- it describes the change of the coupling between an (inverse) scale $L$ and an an (inverse) scale $sL$, for $s$ integer (typically $s=2$)

\[ \bar{g}^2(L) = u \quad \bar{g}^2(sL) = u' \]

$\sigma(s, u) = u'$

- this is a discrete form of the Callan-Symanzik beta function
- differentiate both sides w.r.t. $\mu$ $d/d\mu = -L$ $d/dL$ and use above

\[ \beta[\sqrt{\sigma(s, u)}] = \beta[\sqrt{u}] \sqrt{\frac{u}{\sigma(s, u)}} \frac{d\sigma(s, u)}{du} \]

- so if we know the ssf, we can reconstruct the Callan-Symanzik function recursively
- the step scaling function in PT is given by

\[ \sigma(s, u) = u + 2b_0 \ln(s) u^2 + \cdots \]
Step scaling function

- we next define (in the continuum) a discrete version of the $\beta$-function, the step scaling function
- it describes the change of the coupling between an (inverse) scale $L$ and an (inverse) scale $sL$, for $s$ integer (typically $s=2$)

$$\bar{g}^2(L) = u \quad \bar{g}^2(sL) = u' \quad \sigma(s, u) = u'$$

- this setup is suitable for a NP computation of the coupling / step scaling function
- in practice we compute NP-ly the step scaling function in a range of couplings $u_{\text{min}}$ and $u_{\text{max}}$, corresponding to two scales $\mu_{\text{max}}$ and $\mu_{\text{min}}$; so we obtain the RG-running between them
- the two scales are separated by a power of $s$, i.e. $\mu_{\text{max}} = s^k \mu_{\text{min}}$, typically $s=2$

- the gauge coupling and step scaling function calculations requires choosing a regularization: lattice is the obvious choice
- on the lattice it has an additional dependence on the lattice resolution $L/a$

$$\Sigma(s, u, a/L) = u' \quad \sigma(s, u) = \lim_{a \to 0} \Sigma(s, u, a/L)$$
Step scaling function

- lattice gauge action of choice is the Wilson plaquette one, with some care at the t-boundaries
- lattice fermion action of choice is Wilson, with some care at the t-boundaries
- proceed as follows:

\[ \bar{g}^2(L) = \left[ \frac{\partial \Gamma_0}{\partial \eta} / \frac{\partial \Gamma}{\partial \eta} \right]_{\eta=0} \]

★ choose a lattice with \( L/a \) points in each direction
★ tune bare coupling so that the renormalized coupling has a fixed value
★ at the same bare coupling, compute the renormalized coupling on a lattice twice as big \( 2L/a \)
★ repeat this for several resolutions \( L'/a, L''/a \)
★ extrapolate to the continuum

\[
g_0^2 \rightarrow \bar{g}^2(L) = u
\]
\[
g_0^2 \rightarrow \bar{g}^2(2L) = u'
\]
\[
 u' = \Sigma(2, u, a/L)
\]
\[
\sigma(s, u) = \lim_{a \to 0} \Sigma(s, u, a/L)
\]
Step scaling function: results for $N_f = 2$

\[ g_0^2 \rightarrow \bar{g}^2(L) = u \]
\[ g_0^2 \rightarrow \bar{g}^2(2L) = u' \]
\[ u' = \Sigma(2, u, a/L) \]
\[ \sigma(s, u) = \lim_{a \to 0} \Sigma(s, u, a/L) \]
Step scaling function: results for $N_f = 2$

$$g_0^2 \rightarrow \bar{g}^2(L) = u$$

$$g_0^2 \rightarrow \bar{g}^2(2L) = u'$$

$$u' = \Sigma(2, u, a/L)$$

$$\sigma(s, u) = \lim_{a \rightarrow 0} \Sigma(s, u, a/L)$$

- an expression of the continuum ssf $\sigma(u)$, as a function of the coupling $u$, is obtained by fitting the points above; so we know the ssf in a range $[u_{\text{min}}, u_{\text{max}}]$, corresponding to a range of (still unknown!) scales $[\mu_{\text{max}}, \mu_{\text{min}}]$ (or equivalently $[L_{\text{min}}, L_{\text{max}}]$)

- NB: the agreement/disagreement between PT/NP is a scheme-dependent observation
Step scaling function: results for $N_f = 2$

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- NB: the agreement between PT/NP at low couplings is scheme dependent!!
Step scaling function: results for $N_f = 2$

$g_0^2 \rightarrow \bar{g}^2(L) = u$
$g_0^2 \rightarrow \bar{g}^2(2L) = u'$
$u' = \Sigma(2, u, a/L)$

$\sigma(s, u) = \lim_{a \to 0} \Sigma(s, u, a/L)$

- an expression of the continuum ssf $\sigma(u)$, as a function of the coupling $u$, is obtained by fitting the points above; so we know the ssf in a range $[u_{\text{min}}, u_{\text{max}}]$, corresponding to a range of (still unknown!) scales $[\mu_{\text{max}}, \mu_{\text{min}}]$ (or equivalently $[L_{\text{min}}, L_{\text{max}}]$)

- NB: the agreement/disagreement between PT/NP is a scheme-dependent observation
Gauge coupling: results for $N_f = 2$

- knowing NPly ssf $\sigma(u)$, we can now compute NP-ly the running strong coupling:
- on the previous plot of $\sigma(u)$ vs. $u$, choose a number of discrete couplings:

\[
u_1 = \bar{g}^2(L_{\text{min}}) \leftrightarrow \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}} = \frac{\mu_{\text{max}}}{\mu_{\text{max}}/2} = \frac{\mu_{\text{max}}/2}{\mu_{\text{max}}/4} = \frac{\mu_{\text{max}}/4}{\mu_{\text{max}}/2} = \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}/2} \quad \text{known from PT} \]

\[
u_2 = \bar{g}^2(2L_{\text{min}}) \leftrightarrow \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}/2} = \frac{\mu_{\text{max}}/2}{\mu_{\text{max}}/2} = \frac{\mu_{\text{max}}/2}{\mu_{\text{max}}/4} = \frac{\mu_{\text{max}}/4}{\mu_{\text{max}}/2} = \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}/4} \]

\[
u_3 = \bar{g}^2(4L_{\text{min}}) \leftrightarrow \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}/4} = \frac{\mu_{\text{max}}/4}{\mu_{\text{max}}/2} = \frac{\mu_{\text{max}}/2}{\mu_{\text{max}}/4} = \frac{\mu_{\text{max}}}{\mu_{\text{max}}/2} = \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}/2} \]

\[
u_k = \bar{g}^2(2^k L_{\text{min}} = L_{\text{max}}) \leftrightarrow \frac{\Lambda_{\text{SF}}}{\mu_{\text{min}}} = \frac{2\mu_{\text{min}}}{\mu_{\text{min}}} = \frac{4\mu_{\text{min}}}{\mu_{\text{min}}} = \frac{\mu_{\text{max}}/2}{\mu_{\text{max}}/4} = \frac{\mu_{\text{max}}}{\mu_{\text{max}}/2} = \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}/2} \]

\[
\frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}} = \exp \left[ -\frac{1}{2b_0 \bar{g}^2(\mu_{\text{max}})} \right] \left[ b_0 \bar{g}^2(\mu_{\text{max}}) \right]^{-b_1/(2b_0^2)} \exp \left[ -\int_0^{g(\mu_{\text{max}})} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right]
\]
Gauge coupling: results for $N_f = 2$

- knowing NPls ssf $\sigma(u)$, we can now compute NP-ly the running strong coupling:

- on the previous plot of $\sigma(u)$ vs. $u$, choose a number of discrete couplings:

\[
\begin{align*}
    u_1 &= g^2(L_{\text{min}}) \leftrightarrow \frac{\Lambda_{SF}}{\mu_{\text{max}}} \\
    u_2 &= g^2(2L_{\text{min}}) \leftrightarrow \frac{\Lambda_{SF}}{\mu_{\text{max}}/2} = \frac{\mu_{\text{max}}}{2} \frac{\Lambda_{SF}}{\mu_{\text{max}}} \\
    u_3 &= g^2(4L_{\text{min}}) \leftrightarrow \frac{\Lambda_{SF}}{\mu_{\text{max}}/4} = \frac{\mu_{\text{max}}/2}{4} \frac{\mu_{\text{max}}/2}{\mu_{\text{max}}} \frac{\Lambda_{SF}}{\mu_{\text{max}}} \\
    \vdots & \quad \vdots \\
    u_k &= g^2(2^k L_{\text{min}} = L_{\text{max}}) \leftrightarrow \frac{\Lambda_{SF}}{\mu_{\min}} = \frac{2\mu_{\text{min}}}{\mu_{\text{min}}} \frac{4\mu_{\text{min}}}{\mu_{\text{min}}} \ldots \frac{\mu_{\text{max}}/2}{\mu_{\text{max}}/4} \frac{\mu_{\text{max}}}{\mu_{\text{max}}} \frac{\Lambda_{SF}}{\mu_{\text{min}}/2}
\end{align*}
\]

Known from PT

Thus we obtain the correspondence between $u(L)$ and $\Lambda_{SF}/\mu$ (with $\mu = 1/L$) for the whole range of scales $\mu$.

Iteratively work out couplings $u(L)$ and $u(2L)$ for each pair of successive scales $\mu$ and $\mu/2$ from ssf $\sigma(u)$. 
Gauge coupling: results for $N_f = 2$

- knowing NPls ssf $\sigma(u)$, we can now compute NP-ly the running strong coupling:

- on the previous plot of $\sigma(u)$ vs. $u$, choose a number of discrete couplings:

\[
\begin{align*}
    u_1 &= g^2(L_{\text{min}}) \leftrightarrow \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}} \\
    u_2 &= g^2(2L_{\text{min}}) \leftrightarrow \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}/2} = \frac{\mu_{\text{max}}}{\mu_{\text{max}}/2} \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}/2} \\
    u_3 &= g^2(4L_{\text{min}}) \leftrightarrow \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}/4} = \frac{\mu_{\text{max}}/2}{\mu_{\text{max}}/4} \frac{\mu_{\text{max}}}{\mu_{\text{max}}/2} \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}} \\
    \vdots \quad \vdots \\
    u_k &= g^2(2^k L_{\text{min}} = L_{\text{max}}) \leftrightarrow \frac{\Lambda_{\text{SF}}}{\mu_{\text{min}}} = \frac{2\mu_{\text{min}}}{\mu_{\text{min}}} \frac{4\mu_{\text{min}}}{2\mu_{\text{min}}} \ldots \frac{\mu_{\text{max}}/2}{\mu_{\text{max}}/4} \frac{\mu_{\text{max}}}{\mu_{\text{max}}/2} \frac{\Lambda_{\text{SF}}}{\mu_{\text{max}}}
\end{align*}
\]

known from PT


- NB: again the scale $\mu$ is expressed in units of the (still unknown) $\Lambda_{\text{SF}}$; we need to know $\mu$ in physical units e.g. GeV
all results obtained so far are “purely field theoretic”; i.e. they have been obtained from
the massless QCD action, without any external (experimental) input
this is the reason that everything so far involved dimensionless quantities
in order to make contact with the real world, we need to know $\mu_{min}$ (or $\Lambda_{SF}$) in
physical units
strategic:
- for a series of lattice resolutions $L_{max}/a, L_{max}/a', L_{max}/a'', \ldots$, tune the bare couplings
  $g_0, g_0', g_0'', \ldots$ so as to have the same fixed renormalized coupling $g_R(L_{max}) = \text{const.}$
- for these bare couplings compute some suitable physical quantity; e.g. the proton
  mass $a m_p, a' m_p, a'' m_p, \ldots$ the products $[L_{max}/a] \times [a m_p]$, extrapolated to the continuum for all lattice spacings
  $a, a', a'', \ldots$, gives us $L_{max} m_p$
- use the physical (expt. lly known) value of $m_p$ to get $L_{max}$ (i.e. $\mu_{min}$) and thus $\Lambda_{SF}$
- for historical (quenching) and practical reasons, another observable known as the
  Sommer parameter $r_0$ is used instead of $m_p$
the parameter $r_0$ is the physical distance at which the static quark-antiquark potential $F(r)$ has a chosen fixed value:

$$\left[ r^2 F(r) \right]_{r=r_0} = 1.65$$

phenomenological models suggest that for $\left[ r^2 F(r) \right] = 1.65$, we get $r_0 = 0.5$ fm

the rest is similar to the procedure described previously, based on the proton mass; instead of $m_p$, we have $l/r_0$

so we are in a position to compute $\mu_{\text{min}}$ (or $\Lambda_{\text{SF}}$) in physical units

however, people prefer to see $\Lambda_{\text{MS}}$

this implies that we have to match the SF scheme to $\overline{\text{MS}}$
\( \Lambda \)-dependence of renormalization scheme

- given two schemes “1” and “2”, the corresponding renormalized couplings are connected, to all orders in PT by the relation
  \[
  \bar{g}_1^2 = \bar{g}_2^2 \left[ 1 + c_1 \bar{g}_2^2 + c_2 \bar{g}_2^4 + c_3 \bar{g}_2^6 \cdots \right]
  \]

- recall that the corresponding \( \Lambda \) parameters are written as
  \[
  \Lambda_{1,2} = \lim_{\mu_0 \to \infty} \mu_0 \exp \left[ - \frac{1}{2b_0 \bar{g}_{1,2}^2(\mu_0)} \right] \left[ b_0 \bar{g}_{1,2}^2(\mu_0) \right]^{-b_1/(2b_0^2)}
  \]

- from these expressions we can work out the ratio, valid to all orders in PT
  \[
  \frac{\Lambda_1}{\Lambda_2} = \exp \left[ \frac{c_1}{2b_0} \right]
  \]

- NB: only the first perturbative coefficient is necessary!!

- the scheme matching has been worked out between SF and \( \overline{\text{MS}} \)


\[
\Lambda_{N_f=2}^{\overline{\text{MS}}} = 245(16)(16)\text{MeV} \quad \text{with } r_0 = 0.5\text{fm}
\]
Schrödinger Functional renormalization scheme: quark mass
having dealt with the gauge coupling we turn to the other QCD fundamental parameters, i.e. the quark masses

they are “unphysical” (i.e. non-observable) field theoretic quantities, which depend on the renormalization scale

their RG-running is governed by the anomalous dimension $\gamma$

in a mass independent scheme, $\gamma(g_R)$ depends on the number of flavours but not on the quark masses

it is defined as:

and has the following perturbative expansion:

$$m_R \gamma(g_R) = \mu \frac{\partial m_R}{\partial \mu}$$

$$\gamma(g) = -g^2 \left[ d_0 + d_1 g^2 + d_2 g^4 + \cdots \right]$$

$$d_0 = \frac{8}{(4\pi)^2} \text{ universal}$$

renormalization scheme dependent
Quark mass RG-running and the SF

- the quark mass RG equation is integrated between a minimum and a maximal energy scale; the former is taken to infinity (i.e. coupling to zero)

- this procedure is similar to that exposed in detail for the gauge coupling, and gives rise to a constant quantity, with the dimensions of mass

\[
M_{\text{RGI}} \equiv \lim_{\mu_0 \to \infty} m_R(\mu_0) \left[ 2b_0 \ g_R^2(\mu_0) \right]^{-d_0/(2b_0)}
\]

\[
M_{\text{RGI}} = m_R(\mu) \left[ 2b_0 \ g_R^2(\mu) \right]^{-d_0/(2b_0)} \exp \left[ -\int_0^{g_R(\mu)} dg \left[ \frac{\gamma(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right]
\]

regular in the limit \( g_R(\mu_0) \to 0 \)

- the ratio of the RGI mass \( M_{\text{RGI}} \) to the renormalized mass \( m_R(\mu) \) is a field theoretic quantity, independent of any physical input

- it depends on the flavour number, but not on the quark mass value

- using the definition of the RGI mass for two distinct schemes, it can be shown that it is a scheme independent quantity
Quark mass RG-running and the SF

- the definition of the quark mass step scaling function is the ratio of the renormalized masses at two consecutive scales, at

\[ \sigma_P(s, u) = \frac{m_R(\mu)}{m_R(\mu/s)} = \frac{Z_P^{-1}(aL) m_0(g_0)}{Z_P^{-1}(asL) m_0(g_0)} = \frac{Z_P^{-1}(aL)}{Z_P^{-1}(asL)} \]

at same renorm. coupling \( u \)

\( g_0^2 \) corresponding to \( u \)

this is how it is computed

- computation performed at zero quark mass (i.e. ssf defined in the chiral limit)

- it follows recursive logic of the coupling ssf computation

- the lattice ssf \( \Sigma_P(u, L) \) is computed at several renormalized couplings and extrapolated to the continuum limit

- the \( N_f = 2 \) result is shown
Quark mass RG-running and the SF

- knowing NPlly ssf $\sigma p(u)$, we can now compute NP-ly the running strong coupling

\[
\frac{M}{m_R(\mu_{\text{min}})} = \frac{m_R(2\mu_{\text{min}})}{m_R(\mu_{\text{min}})} \frac{m_R(4\mu_{\text{min}})}{m_R(2\mu_{\text{min}})} \ldots \frac{m_R(\mu_{\text{max}}/2)}{m_R(\mu_{\text{max}}/4)} \frac{m_R(\mu_{\text{max}})}{m_R(\mu_{\text{max}}/2)} \frac{M}{m_R(\mu_{\text{max}})}
\]

known from ssf $\sigma p(u)$

\[
\frac{M}{m_R(\mu_{\text{max}})} = \left[2b_0\bar{g}^2(\mu_{\text{max}})\right]^{-d_0/(2b_0^2)} \exp\left[\int_0^{\bar{g}(\mu_{\text{max}})} dg \left[\frac{\gamma(g)}{\beta(g)} - \frac{d_0}{b_0 g}\right]\right]
\]

\[
\frac{M}{m_R(\mu_{\text{min}})} = 1.297(16) \quad N_f = 2
\]

Quark mass RG-running and the SF

- now the RGI quark mass of a given flavour $f$ can be computed

$$M_f = \frac{M_f}{m_R(\mu_{\text{min}})} m_R(\mu_{\text{min}}) = \frac{M_f}{m_R(\mu_{\text{min}})} \lim_{\alpha \to 0} Z_P^{-1}(a\mu_{\text{min}}, g_0) m_{\text{PCAC}}(g_0)$$

- the bare PCAC quark mass is defined as

$$m_{\text{PCAC}} = \frac{Z_A \partial_0 f_A}{2 f_P}$$

- for SF correlation functions:

$$f_P = \langle P(x) \ O(0) \rangle$$
$$f_A = \langle A_0(x) \ O(0) \rangle$$

scheme independent
scheme dependent
scale dependent
regularization dependent
scale dependent
regularization dependent
scale dependent
known in the C.L.
must be computed NPly
flavour dependence

boundary source composite field with pseudoscalar quantum numbers
Quark mass RG-running and the SF

- now the RGI quark mass of a given flavour $f$ can be computed

\[ M_f = \frac{M_f}{m_{R}(\mu_{\text{min}})} \quad m_{R}(\mu_{\text{min}}) = \frac{M_f}{m_{R}(\mu_{\text{min}})} \lim_{a \to 0} Z_{P}^{-1}(a\mu_{\text{min}}, g_{0}) \, m_{\text{PCAC}}(g_{0}) \]

- the SF renormalization condition for the pseudoscalar density is:

\[ \frac{Z_{P}(L_{\text{max}})}{\sqrt{f_{1}}} \frac{f_{P}(L_{\text{max}}/2)}{\sqrt{f_{1}}} = \text{T.L.} \left[ \frac{f_{P}(L_{\text{max}}/2)}{\sqrt{f_{1}}} \right] \]

- cancels boundary quark field renormalization
- must be computed NPly
- flavour dependence

known in the C.L.

scheme independent

scheme dependent

scale dependent

regularization dependent

scale dependent

regularization dependent

scale dependent
Quark mass RG-running and the SF

- now the RGI quark mass of a given flavour $f$ can be computed

$$M_f = \frac{M_f}{m_R(\mu_{\text{min}})} m_R(\mu_{\text{min}}) = \frac{M_f}{m_R(\mu_{\text{min}})} \lim_{a \to 0} Z_P^{-1}(a\mu_{\text{min}}, g_0) m_{\text{PCAC}}(g_0)$$

- simulations at the physical up/down quark masses are a daunting task
- simulations in the mass range $[m_s/4, m_c]$ are nowadays feasible
- a nice approach is to define a reference quark mass (approximately $m_s/2$) for which a “Kaon” consisting of two degenerate valence quarks weighs 495 MeV (the “physical” value)
- this “world” is a two degenerate flavour ($N_f = 2$) theory
- the previous SF procedure, once the bare quark mass is tuned to the reference quark mass etc., gives $M_{\text{ref}} = 72 (3) (13)$ MeV
- next use the chiral PT result $M_s = 48/25 M_{\text{ref}}$, to obtain $M_{\text{strange}} = 138 (5) (26)$ MeV

Recapitulation of RG-running with the SF

PT-regime: compute RGI quantities ($\Lambda_{QCD}$, $M_{RGI}$); use them in high-energy phenomena (jet Physics)

$\mu = 2^n / L_{max}$

$n$ recursive steps

Schrödinger Functional scheme

renormalization scale (energy)

hadronic scheme

$g_R \leftrightarrow m_p^{phys}$
$m_{u/d}^R \leftrightarrow m_{\pi}^{phys}$

NP-regime: compute hadronic matrix elements and SF renorm. constants

$L_{max} \sim 0.5 \text{fm}$